SEMICONTINUITY OF COMPOSITE FUNCTION VIA THE INHERITANCE MECHANISM ON THE SEMICONTINUITY OF SET-VALUED MAPS

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A composite function is a function which is the nesting of two or more functions to form a single new function. Such operation frequently preserves several mathematical properties of each nested function. For instance, a composition of continuous maps is continuous on topological spaces. From the view point of vector optimization and set optimization, this kind of inheritance by composite operations is important and useful to prove extended results and to get characterizations of optimal solutions through scalarization. This is a typical approach by which optimization problems with vector-valued or set-valued maps can be easily handled by converting vectors or sets into real numbers; see [3] and [4, 5].

Let X, Y, and C be a topological space, an ordered real topological vector space, and a convex ordering cone in Y, respectively. In 1996, Tanaka [12] proved the inheritation of the composition of a cone-continuous vector-valued function $f : X \to Y$ and any element of the dual cone $C^+_{X'}$, which consists of nonnegative continuous linear functions on a convex cone $C \subset X$. In 2006, Kimura and Tanaka [7] mentioned its generalized inheritation of the composition of a cone-continuous vector-valued function $f : X \to Y$ and the Gerstewitz's (Tammer's) nonlinear scalarizing functional:

$$h_C(v; d) := \inf \{t \in \mathbb{R} \mid v \in td - C\} \tag{1}$$

where C is a convex cone in a real vector space and $d \in C$. This scalarizing functional $h_C(\cdot; d)$ is sublinear (i.e., $h_C(v_1 + v_2; d) \leq h_C(v_1; d) + h_C(v_2; d)$ and $h_C(tv; d) = th_C(v; d)$ for $t > 0$) and hence this conversion is called "sublinear scalarization," found early in [2, 10]. If convex cone C is a half space, that is, $C = \{v \mid \langle w, v\rangle \geq 0\}$ with w satisfying $\langle w, w\rangle = 1$, then $h_C(v; w) = \langle w, v\rangle$. Therefore this special functional is a certain generalization of linear scalarization including the notions of weighted sum and inner product. Accordingly, this idea has inspired some researchers to develop particular scalarization methods for sets.

Recently, Ike, Liu, Ogata and Tanaka [6] show certain results on the inheritance property of some kinds of continuity of set-valued maps via scalarization functions for sets: if a set-valued map has a kind of continuity (lower continuity or upper continuity; see [4]) then the composition of its set-valued map and a certain scalarization function assures a similar semicontinuity to its scalarization function defined on the family of nonempty subsets of a real topological vector space. Their results are generalizations of results in earlier study by Kuwano, Tanaka and Yamada [9]. However, the statements of inheritance in [6] are confined to four types out of the six set-relations proposed by Kuroiwa, Tanaka and Ha [8]. On the other hand, Sonda, Kuwano, and Tanaka [11] introduce two kinds of continuity with respect to cone, called "cone continuity," for set-valued maps by analogy with semicontinuity for real-valued functions, and they investigate the inheritance properties on cone continuity of parent set-valued maps via scalarization. Therefore, it is interesting to investigate the inheritance of cone continuity for set-valued maps via general scalarization functions for sets in the same manner as [6].

The aim of this talk is to introduce the mechanism by which composite functions of a set-valued map and a scalarization function transmit semicontinuity of parent set-valued maps through several scalarization for sets; see [1].

References

[1] P. Dechboon and T. Tanaka, *Inheritance properties on cone continuity for set-valued maps via scalarization*, Minimax Theory Appl., **9** (2024), 201–224.

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[2] C. Gerstewitz, (C. Tammer), *Nichtkonvexe dualität in der vektoroptimierung*, Wiss. Z. Tech. Hochsch. Leuna-Merseburg **25**(3) (1983), 357–364.

[3] C. Gerth (Tammer) and P. Weidner, *Nonconvex separation theorems and some applications in vector optimization*, J. Optim. Theory Appl., **67** (1990), 297–320.

[4] A. Göpfert, H. Riahi, C. Tammer, and C. Zălinescu, *Variational methods in partially ordered spaces*, vol. 17 of CMS Books in Mathematics, Springer-Verlag, New York, 2003.

[5] A. Hamel and A. Löhne, *Minimal element theorems and Ekeland's principle with set relations*, J. Nonlinear Convex Anal., **7** (2006), 19–37.

[6] K. Ike, M. Liu, Y. Ogata, and T. Tanaka, *Semicontinuity of the composition of set-valued map and scalarization function for sets*, J. Appl. Numer. Optim., **1** (2019), 267–276.

[7] K. Kimura and T. Tanaka, *Existence theorem of cone saddle-points applying a nonlinear scalarization*, Taiwanese J. Math., **10 (2)** (2006), 563–571.

[8] D. Kuroiwa, T. Tanaka, and T. X. D. Ha, *On cone convexity of set-valued maps*, in Proceedings of the Second World Congress of Nonlinear Analysts, Part 3 (Athens, 1996), Nonlinear Anal., **30** (1997), 1487–1496.

[9] I. Kuwano, T. Tanaka, and S. Yamada, *Unified scalarization for sets and set-valued Ky Fan minimax inequality*, J. Nonlinear Convex Anal., **11** (2010), 513–525.

[10] A. M. Rubinov, *Sublinear operators and their applications*, Russian Math. Surveys. **32** (1977), 115–175.

[11] Y. Sonda, I. Kuwano, and T. Tanaka, *Cone-semicontinuity of set-valued maps by analogy with real-valued semicontinuity*, Nihonkai Math. J., **21** (2010), 91–103.

[12] T. Tanaka, *Generalized semicontinuity and existence theorems for cone saddle points*, Appl. Math. Optim, **36(3)** (1997), 313–322.

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