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MATRIX OPTIMIZATION PROBLEMS WITH MAXIMUM-NORM CONSTRAINTS AND THEIR APPLICATION TO FINITE ELEMENT INTERPOLATION ERROR ESTIMATES

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Matrix optimization problems with maximum-norm constraints appear naturally when one tries to obtain explicit and sharp constants in finite element error analysis. Based on the work of Galindo, Ike, and Liu [1], this talk discusses such a problem arising from the maximum-norm estimate for the linear Lagrange interpolation on a triangular element K. The target constant $C^L(K)$ is defined by

$$\left\|u - \Pi^L u\right\|_{\infty,K} \le C^L(K)|u|_{2,K}, \qquad u \in H^2(K),$$

where $\Pi^L u$ denotes the linear Lagrange interpolation of u. Unlike the classical L^2- or H^1-error estimates, the use of the maximum norm in the left-hand side leads to a nonsmooth constraint that is difficult to treat directly.

Let

$$\lambda(K) := \inf_{u \in V^L(K)} \frac{|u|_{2,K}^2}{\|u\|_{\infty,K}^2}, \qquad V^L(K) := \{u \in H^2(K) \mid u(p_i) = 0,\ i = 1,2,3\}.$$

Then $C^L(K) = \lambda(K)^{-1/2}$. Using the Fujino–Morley finite element space $V_h^{\mathrm{FM}}(K)$, the corresponding finite dimensional problem becomes

$$\lambda_h = \min_{x \in \mathbb{R}^M} x^T A x, \qquad \text{subject to } \left\|\sum_{i=1}^{M} x_i \phi_i\right\|_{\infty,K} \ge 1,$$

where $A = (\langle \phi_i, \phi_j \rangle_h)$ is the stiffness matrix associated with the broken H^2 inner product.

To handle this constraint efficiently, we use the Bernstein representation of piecewise quadratic functions. The convex-hull property of Bernstein polynomials gives

$$\left\|\sum_{i=1}^{M} x_i \phi_i\right\|_{\infty,K} \le \|Bx\|_\infty,$$

where B is the matrix mapping Fujino–Morley coefficients to Bernstein coefficients. This leads to the relaxed matrix optimization problem

$$\lambda_{h,B} = \min_{x \in \mathbb{R}^M} x^T A x, \qquad \text{subject to } \|Bx\|_\infty \ge 1.$$

Although this is still a nonconvex constrained problem, its optimal value can be obtained explicitly. Since A is positive definite, by taking $A = R^T R$ and putting $y = Rx$, the problem becomes

$$\lambda_{h,B} = \min_{y \in \mathbb{R}^M} y^T y, \qquad \text{subject to } \left\|BR^{-1}y\right\|_\infty \ge 1.$$

If b_i^T denotes the ith row of BR^{-1}, then

$$\lambda_{h,B} = \frac{1}{\max_i \|b_i\|_2^2}.$$

Equivalently, without explicitly computing the Cholesky factor, one has

$$\lambda_{h,B} = \frac{1}{\max(\mathrm{diag}(BA^{-1}B^T))}.$$

Thus the constrained optimization problem is converted into a matrix computation involving the diagonal entries of $BA^{-1}B^T$. Hence no nonlinear iterative optimization is required; the computation is reduced to sparse matrix assembly and linear solves. On the other hand, for

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dense finite element meshes the dimensions of A and B become large, and the efficient evaluation of the required diagonal entries remains an important computational issue. As future work, we will improve the present method for large-scale matrix computations, especially for refined finite element meshes.

We also explain how this computable lower bound for λ_h gives a rigorous lower bound for $\lambda(K)$, and hence a rigorous upper bound for $C^L(K)$, by using the orthogonality of the Fujino–Morley projection. For uniform meshes, a sharper correction formula can be derived from scaling properties of similar triangles. Numerical examples show that the proposed approach gives sharp and efficient estimates. For instance, for the unit right isosceles triangle, the method yields

$$0.40432 \leq C^L(K) \leq 0.41596,$$

while a direct theoretical estimate gives only the much rougher upper bound 1.3712. Interval arithmetic can also be incorporated in the assembly and matrix evaluation steps; for mesh size 1/64, the validated upper bound is contained in

$$[\,0.4159516728,\ 0.4159516793\,].$$

These results indicate that maximum-norm constrained matrix optimization, combined with finite element structure and Bernstein polynomial techniques, provides an effective framework for rigorous interpolation error estimation.

References

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