Real Numbers II

Class: ______ Number:______ Name:____________

1. Algebra Formulas

i) Cube of a sum: $(a+b)^3 =$ **;** Cube of a difference: $(a-b)^3 =$

ii) Sum of cubes: $a^3+b^3 =$ **;** Difference of cubes: $a^3-b^3 =$

Example 1 Expand the following expressions (1) $(a+2b)^3$ (2) $(2a-1)^3$.

Solution

Example 2 Factorize the following expressions (1) $x^3-12x^2+48x-64$ (2) $27x^3-1$.

Solution

2. Nested Radicals

i) $\sqrt{(a+b)+2\sqrt{ab}} =$ **ii)** $\sqrt{(a+b)-2\sqrt{ab}} =$

Example Simplify the following radicals (1) $\sqrt{8-2\sqrt{15}}$ (2) $\sqrt{7+\sqrt{24}}$.

Solution

3. AM-GM Inequality

If a and b are non-negative real numbers, then

$$\frac{a+b}{2} \geq \sqrt{ab}\,.$$

The equality holds if and only if $a=b$.

Example Let a and b be two positive real numbers

(1) Given $2a+3b=12$, find the maximum value of ab and the ordered pair (a, b) for which it occurs.

(2) Given $ab=10$, find the minimum value of $5a+2b$ and the ordered pair (a, b) for which it occurs.

Solution

4. Given that $a=6$, $b=\frac{20}{3}$, $c=2\sqrt{10}$, and d, where d is a rational number, mark these four numbers on the number line, namely $A(a)$, $B(b)$, $C(c)$, and $D(d)$. Which of the following statements is true?

A) $a+b+c+d$ is a rational number

B) $abcd$ is an irrational number

C) It is possible that the distance between point D and point C is equal to $2\sqrt{10}+6$

D) The midpoint of point A and point B is located at the right side of point C

E) There are 14 positive integers and 1 negative integer on the number line that are within a distance of less than 8 from point B.

Solution

Real Numbers II

Warm Up

Hi everyone, please go back to your seats and take out your worksheet. Today's topic is still real numbers, and let's review what we've learned this week. We have learned some algebra formulas, the simplest form of radicals, nested radicals, and the AM-GM inequality. Let's look at part 1 of the worksheet.

Vocabulary

1. Cube of a sum(和立方) 2. Cube of a difference(差立方) 3. Sum of cubes(立方和) 4. Difference of cubes(立方差) 5. nested radical(多重根式) 6. semicircle (半圓) 7. diameter(直徑) 8. segment(線段) 9. perpendicular(垂直的) 10. AM-GM inequality(算術-幾何不等式，AM: Arithmetic Mean 算術平均數 Geometric Mean 幾何平均數) 11. maximum(最大值) 12. minimum(最小值) 13. midpoint(中點).

Illustrations

Part 1.

Let's look at part 1. Does anyone know what the term for "**Cube of a sum**[1]" is in Chinese? (Student's name), can you tell us? Yes, it's "和立方," which means we first add *a* and *b*, then cube the result. The one beside it, "**Cube of a difference**[2]," is "差立方," which means we first subtract *a* from *b*, then cube the result.

The following lines are "**Sum of cubes**[3]" and "**Difference of cubes**[4]," which are "立方和" and "立方差," respectively. This means we first cube a and b, then add *a* and *b* or subtract *a* from *b*.

Now, I'll give you 2 minutes to fill in all the formulas. (After two minutes)

i) (Student's name), can you read the formulas for "cube of a sum" and "cube of a difference" for us?

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

(The cube of the sum of *a* and *b* **equals** *a* cubed **plus** 3 times *a* squared times *b* **plus** 3 times *a* times *b* squared **plus** *b* cubed.)

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

(The cube of the difference of *a* and *b* **equals** *a* cubed **minus** 3 times *a* squared times *b* **plus** 3 times *a* times *b* squared **minus** *b* cubed.)

ii) (Student's name), can you read the formulas for "Sum of cubes" and "Difference of cubes" for us?

$$a^3+b^3=(a+b)\left(a^2-ab+b^2\right)$$

(The sum of the cubes of *a* and *b* **equals** the sum of *a* and *b* **times** a squared **minus** *a* times *b* **plus** b squared.)

$$a^3-b^3=(a-b)\left(a^2+ab+b^2\right)$$

(The difference of the cubes of *a* and *b* **equals** the difference of *a* and *b* **times** a squared **plus** *a* times *b* **plus** b squared.)

Perfect! Let's look at the following examples.

Example 1 Expand the following expressions (1) $(a+2b)^3$ (2) $(2a-1)^3$.

Solution

(1) To expand the following expressions, we can apply the formula of the cube of a sum. Substituting *a* for *a* and 2*b* for *b*, then we get

$$\begin{aligned}(a+2b)^3&=\left(a+(2b)\right)^3\\&=a^3+3(a)^2(2b)+3(a)(2b)^2+(2b)^3\\&=a^3+6a^2b+12ab^2+8b^3.\end{aligned}$$

(2) Using the method from question (1), we can apply the formula of the cube of a difference. Substituting 2*a* for *a* and 1 for *b*, then we get

$$\begin{aligned}(2a-1)^3&=\left((2a)-(1)\right)^3\\&=(2a)^3-3(2a)^2(1)+3(2a)(1)^2-(1)^3\\&=8a^3-12a^2+6a-1.\end{aligned}$$

Example 2 Factorize the following expressions (1) $x^3+12x^2+48x+64$ (2) $27x^3-1$.

Solution

(1) To factorize the following expressions, we can apply the formula of the sum of cubes. Substituting *x* for *a* and 4 for *b*, then we get

$$x^3+12x^2+48x+64=(x)^3+3(x)^2(4)+3(x)(4)^2+(4)^3$$
$$=(x+4)^3.$$

(2) Using the method from question (1), we can apply the formula of the difference of cubes.

Substituting 3x for *a* and 1 for *b*, then we get

$$27x^3-1=(3x)^3-(1)^3$$
$$=((3x)-(1))((3x)^2+(3x)(1)+(1)^2)$$
$$=(3x-1)(9x^2+3x+1).$$

Part 2. (Student's name), can you tell me what "**nested radical**[5]" is in Chinese?

Yes, it's "多重根式." The nested radical formula is derived from the square of a sum.

$\sqrt{(a+b)+2\sqrt{ab}}=\sqrt{\left(\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2+2\sqrt{a}\times\sqrt{b}}=\sqrt{\left(\sqrt{a}+\sqrt{b}\right)^2}=\sqrt{a}+\sqrt{b}$. (can be read as: square root of *a* plus *b* plus 2 times square root of *a* times *b* **equals** square root of root of *a* squared plus root of *b* squared plus 2 times root of *a* times root of *b* **equals** square root of root of *a* plus root of *b* whole squared **equals** square root of *a* plus square root of *b*.) In general, we let $a \ge b \ge 0$ to ensure that the result of simplifying the nested radical is positive. Here is the proof which is derived from the square of a difference $\sqrt{(a+b)+2\sqrt{ab}}=\sqrt{\left(\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2-2\sqrt{a}\times\sqrt{b}}=\sqrt{\left(\sqrt{a}-\sqrt{b}\right)^2}=\sqrt{a}-\sqrt{b}$.

Nested Radicals

i) $\sqrt{(a+b)+2\sqrt{ab}}=$ **ii)** $\sqrt{(a+b)-2\sqrt{ab}}=$

Example Simplify the following radicals (1) $\sqrt{8-2\sqrt{15}}$ (2) $\sqrt{7+\sqrt{24}}$.

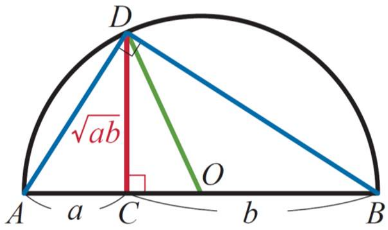
Solution

(1) $\sqrt{8-2\sqrt{15}}=\sqrt{(5+3)-2\sqrt{5}\times\sqrt{3}}=\sqrt{\left(\sqrt{5}-\sqrt{3}\right)^2}=\sqrt{5}-\sqrt{3}$

(2) $\sqrt{7+\sqrt{24}}=\sqrt{7+2\sqrt{6}}=\sqrt{(6+1)+2\sqrt{6}\times\sqrt{1}}=\sqrt{\left(\sqrt{6}+\sqrt{1}\right)^2}=\sqrt{6}+\sqrt{1}=\sqrt{6}+1$.

Part 3. (Student's name), please read part 3 for us.

(After reading) There is an important condition "non-negative," which means that the numbers *a* and *b* can't be negative. Let's have a quick review of the proof. Consider a **semicircle**[6] with a **diameter**[7] AB, and center O. Let C be a point on line **segment**[8] AB, such that $\overline{AC}=a$ and $\overline{CB}=b$. Construct a **perpendicular**[9] line at C of the diameter AB, which intersects semicircle at D. Then we know that segment CD is $\sqrt{ab}$, and radius is $\frac{a+b}{2}$. Let's focus on this right triangle OCD. $\overline{OD}$ is the hypotenuse of ΔOCD, which is the longest side of ΔOCD. Hence, $\overline{OD}>\overline{CD}$, namely $\frac{a+b}{2}>\sqrt{ab}$.



Particularly, when point C is exactly the center O, $a=\overline{AC}=\overline{CB}=b$ and $\overline{OD}=\overline{CD}$, namely $\frac{a+b}{2}=\sqrt{ab}$.

Therefore, we derived the **AM-GM inequality**[10], $\frac{a+b}{2}\geq\sqrt{ab}$ (read as: *a* plus *b* over 2 is greater than or equal to square root of *ab*.)

Example Let *a* and *b* be two positive real numbers

(1) Given $2a+3b=12$, find the **maximum**[11] value of *ab* and the ordered pair (*a*, *b*) for which it occurs.

(2) Given $ab=10$, find the **minimum**[12] value of $5a+2b$ and the ordered pair (*a*, *b*) for which it occurs.

Solution

(1) Using the AM-GM inequality, we have

$$\frac{2a+3b}{2}\geq\sqrt{2a\cdot 3b}\Rightarrow\frac{12}{2}\geq\sqrt{6ab}.$$

Namely, $ab\leq 6$.

Since $ab=6$, $2a=3b=6$, which is $a=3$ and $b=2$.

Therefore, ab has a maximum value of 6 with $(a,b)=(3,2)$.

(2) Using the AM-GM inequality, we have

$$\frac{5a+2b}{2}\geq\sqrt{5a\cdot 2b}=\sqrt{10ab}=10.$$

Namely, $5a+2b\geq 10$.

Since $5a+2b=10$, $5a=2b=5$, which is $a=1$ and $b=\frac{5}{2}$.

Therefore, $5a+2b$ has a minimum value of 10 with $(a,b)=\left(1,\frac{5}{2}\right)$.

Part 4. (Student's name), please read part 4 for us.

(After reading) Let's see the statements accordingly.

A) Since $a+b+c+d=6+\frac{20}{3}+2\sqrt{10}+d$ and d is a rational number, we know that $a+b+c+d$ is a rational number $6+\frac{20}{3}+d$ plus an irrational number $2\sqrt{10}$. Hence, $a+b+c+d$ is an irrational number. This statement is false.

B) Since $abcd=6\cdot\frac{20}{3}\cdot 2\sqrt{10}\cdot d$ and d is a rational number, we know $abcd$ is a rational number $6\cdot\frac{20}{3}\cdot 2\cdot d$ times an irrational number $2\sqrt{10}$. Hence, $abcd$ is an irrational number. This statement is true.

C) The distance between point D and point C is $\left|2\sqrt{10}-d\right|$. If $d=-6$ or $d=4\sqrt{10}+6$, then $\left|2\sqrt{10}-d\right|$ could be $2\sqrt{10}+6$. This statement is true.

D) The midpoint of point A and point B is $\frac{a+b}{2}=\frac{6+\frac{20}{3}}{2}=\frac{19}{3}$, and $\frac{19}{3}=6.\bar{3}$ is greater than $2\sqrt{10}\approx 6.324$. Hence, The **midpoint**[13] of point A and point B is located at the right side of point C. This statement is true.

E) Let the distance between point $X(x)$ and point B be less than 8, then we have $|x-b|<8\Rightarrow\left|x-\frac{20}{3}\right|<8$, namely $-8+\frac{20}{3}<x<8+\frac{20}{3}\Rightarrow-\frac{4}{3}<x<\frac{44}{3}$. Hence, if we only take the integer of x, then x could be $-1,0,1,2,\ldots,14$. Therefore, this statement is true.

References

1. 許志農、黃森山、陳清風、廖森游、董涵冬（2019）。數學 1：單元 1 實數。龍騰文化。
2. Randall I. Charles, Basia Hall, Dan Kennedy, Allan E. Bellman, Sadie Chavis Bragg, William G. Handlin, Stuart J. Murphy, & Grant (2012). *Pearson Algebra 2 Common Core.* Pearson.

製作者：臺北市立陽明高中 吳柏萱 教師