雙語教學主題(國中八年級下學期教材)中垂線與角平分線

Practice/Test: Applications of triangle congruency

Perpendicular bisectors and angle bisectors

Vocabulary

CPCTC stands for corresponding parts of the congruent triangles are congruent		兩全等三角形的對應邊和對應角都對應相等	
Triangle congruence theorem		全等三角形全等性質(定理)	
Isosceles triangle	等腰三角形	two legs	兩腰
base angles	底角	vertex angle	頂角
angle bisector	角平分線	intersect	相交於
reflexive property	反身性(共用邊角)	bisect	平分
perpendicular bisector	垂直平分線	The Pythagorean Theorem	畢氏定理
equilateral triangle	正三角形	Quadrilateral	四邊形
take the square root of both sides	等號兩邊開根號	Simplify the equation	方程式化簡
respectively	按順序地	right triangle	直角三角形
arbitrary point	任意點	intersection point	交點
perimeter	周長	converse theorem	逆定理
trapezoid	梯形	equidistant	等距離的
measure	度量、測量	segment	線段

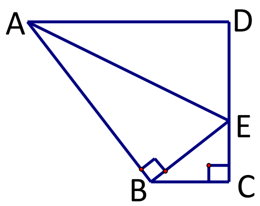
(我個人很喜歡這個 CPCTC 的表示法，否則要寫很多中文字...)

老師們好，這是全等三角形的應用練習—中垂線和角平分線。這份教材的前半是題目加解答，後半是題目，方便老師們參考使用。老師們可以依據學生的需求選取適當的題目供學生做小測或是練習。

Q1:

In trapezoid ABCD, $\overline{AD} // \overline{BC}$, $\angle C = \angle ABE = 90°$, $\overline{AE}$ is the angle bisector of $\angle BAD$.

If $\overline{CD} = 10$, $\overline{BC} = 2\sqrt{5}$, find the measure of $\overline{BE}$.



ANSWER:

We need to find the measure of $\overline{BE}$, we certainly look into $\triangle BCE$.
In $\triangle BCE$, $\angle ABE=90°$, then

$$\overline{BE}^2=\overline{BC}^2+\overline{CE}^2 \quad \cdots\cdots(1)$$

$\overline{BC}$ is given, and $\overline{CE}=\overline{CD}-\overline{DE}=10-\overline{DE}$

But $\overline{DE}=\overline{BE}$ (Property of angle bisector)

Let $\overline{BE}$ be x, then $\overline{DE}=\overline{BE}=x$ and the equation(1)

$$\overline{BE}^2=\overline{BC}^2+\overline{CE}^2$$

Will be

$$x^2=\left(2\sqrt{5}\right)^2+\left(10-\overline{DE}\right)^2$$

$$x^2=\left(2\sqrt{5}\right)^2+\left(10-x\right)^2$$

Simplify the equation,

x=6

That is:

$\overline{BE}=x=6$#

Q2:

$\triangle ABC$ is a right triangle, $\angle C=90°$, $\overline{MN}\perp\overline{AB}$, and

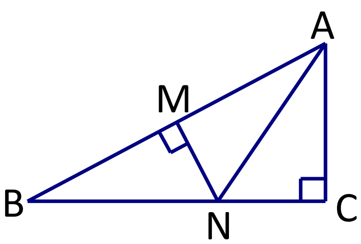
$\overline{AN}=\overline{BN}$

If $\overline{AB}=10$, $\overline{AC}=6$,

Find (1) The length of $\overline{AM}$

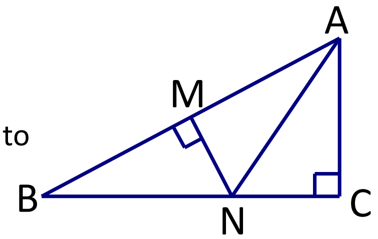
(2) The length of $\overline{AN}$

(3) The area of $\triangle AMN$ (I'll show one solution, students can solve the problem with other methods.)



ANSWER:

Sol:



(1) Because $\overline{MN} \perp \overline{AB}$, and $\overline{AN} = \overline{BN}$, according to the converse statement of perpendicular bisector of a segment, point N is on the perpendicular bisector $\overline{MN}$ of segment AB.

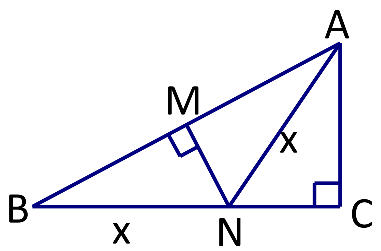
We get

$$\overline{AM} = \overline{BM} = \frac{1}{2}\overline{AB} = 5\#$$

(2) If we want to get the length of $\overline{AN}$, we need to look into the right triangles AMN or ACN. But in ΔAMN, we only know the length of $\overline{AM}$, it's impossible to get two other side lengths of ΔAMN under this situation. On the other hand, in ΔACN, we have some related given information. So let's do it

In the right triangle ABC, $\overline{AB}=10$, $\overline{AC}=6$,

By the Pythagorean theorem,



$$\overline{BC}^2 = \overline{AB}^2 - \overline{AC}^2$$

$$=10^2-6^2$$

$$=64$$

Get $\overline{BC}=8$

Let $\overline{AN} = \overline{BN} = x$ (Property of perpendicular bisector)

Then $\overline{CN} = \overline{BC} - \overline{BN} = 8-x$

In the right ΔACN,

$$\overline{AN}^2 = \overline{AC}^2 + \overline{CN}^2 \quad \text{(The Pythagorean theorem)}$$

$$X^2=6^2+(8-x)^2$$

$$x^2=36+64-16x+x^2$$

$$x=\frac{25}{4}$$

i.e. the length of $\overline{AN} = \frac{25}{4}\#$

(3) According to the information we have, I think the easiest way to get the area of ΔAMN is

The area of $\triangle AMN$

The area of $\triangle AMN=\frac{1}{2}\cdot$the area of $\triangle ABN$

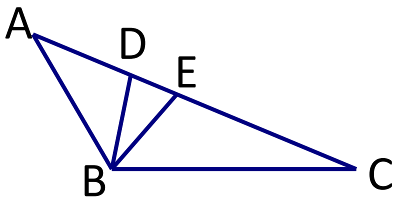
$=\frac{1}{2}$(the area of $\triangle ABC$-the area of $\triangle ACN$)

$=\frac{1}{2}\left(\frac{8\cdot 6}{2}-\frac{\left(8-\frac{25}{4}\right)\cdot 6}{2}\right)$ $\left(\overline{CN}=\overline{BC}-\overline{BN}=8-\frac{25}{4}\right.$

Sorry a little ugly)

$=\frac{75}{8}$ #

Q3:



In $\triangle ABC$, points D and E are on $\overline{AC}$, $\overline{AB}=\overline{AE}$, $\overline{CB}=\overline{CD}$.

If $\angle ABC=120°$, find the measure of $\angle DBE$.

ANSWER

Sol:

Let the measure of $\angle ABD=a$, $\angle CBE=b$, $\angle BDE=\alpha$, $\angle BED=\beta$, and $\angle DBE=x$, as shown in Figure 1.

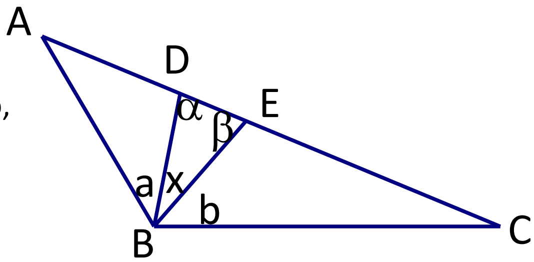


Figure 1

Now let's list all the equations we get here.

$a+x+b=\angle ABC=120°$ (Given)

$\alpha=x+b$ ⋯ (1) (In $\triangle BCD$, $\overline{CB}=\overline{CD}$, angles opposite congruent segments are congruent.)

$\beta=a+x$ ⋯ (2) (Same thing in $\triangle ABE$)

Then In $\triangle BDE$,

$x+\alpha+\beta=180°$ (Triangle interior angle sum theorem)

$x+x+b+a+x=180°$ (Replace α and β from (1) and (2))

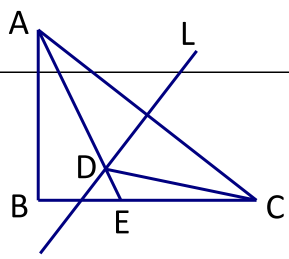
$\Rightarrow x+120°+x=180°$ ($\angle ABC=120°$)

$2x=60°$

$x=30°$ #

Q4:

In $\triangle ABC$, $\angle ABC=90°$, $\overline{AE}$ bisects $\angle BAC$, and



intersects $\overline{BC}$ at point E.

Line L is the perpendicular bisector of $\overline{AC}$, and intersects $\overline{AE}$ at point D.

If $\angle DCE=24°$, find the measure of $\angle CDE$.

ANSWER

Sol:

In $\triangle ABC$,

Let $\angle BAE=\angle CAE=x°$ ($\overline{AE}$ bisects $\angle BAC$)

Then

$\angle ACD=\angle CAD=X°$ (Line L is the perpendicular bisector of $\overline{AC}$)

And $\angle CDE=2x°$ (Triangle exterior angle theorem of $\triangle ACD$)

$\angle AEB=2x°+24$ (Triangle exterior angle theorem of $\triangle CDE$)

In $\triangle ABC$,

$\angle A+\angle B+\angle AEB=180°$ (Triangle interior angle sum theorem)

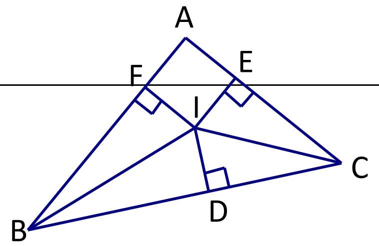
$x°+90°+2x°+24=180°$ (Replace all the information above)

$3x°=66°$

$x=22$#

Q5:

Point I is an interior point of $\triangle ABC$,



Point P is equidistant from $\overline{AB}$, $\overline{BC}$ $\overline{CA}$ respectively.

If ∠A =70°, find the measure of ∠BIC.

ANSWER

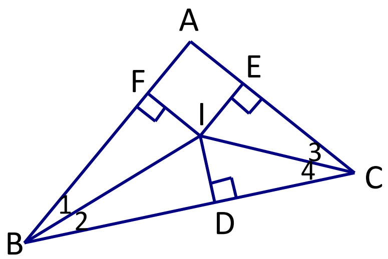
Sol"

We need to apply the converse property of an angle bisector.

Since $\overline{IF}=\overline{ID}$,we know $\overline{BI}$ is the angle bisector of ∠ABC.

Similarly, $\overline{AI}$ and $\overline{CI}$ are angle bisectors of ∠BAC and ∠ACB respectively.

That is: ∠1=∠2, ∠3=∠4, as shown in Figure 1.



Let's go!

In △BIC, ∠BIC+∠2+∠4=180° ⋯ (1)

(Triangle interior angle sum theorem)

In △ABC, ∠A+∠ABC+∠ACB=180°

(Triangle interior angle sum theorem)

i.e. ∠A+2∠2+2∠4=180° ⋯ (2)

2· (1) 2∠BIC+2∠2+2∠4=2· 180° ⋯ (3)

From (3), we get

2· 180°-2∠BIC=2∠2+2∠4

=180°-∠A (From (2))

Simplify the equation, we have

180°+∠A =2∠BIC

$$\angle BIC=90°+\frac{1}{2}\angle A \quad \cdots ★$$

So $$\angle BIC=90°+\frac{1}{2}\cdot 70$$

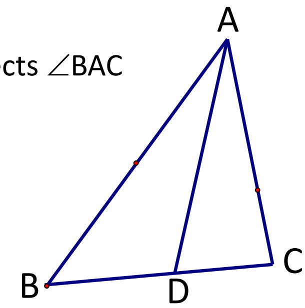
=125° #

The result of the statement ★ is quite commonly seen. Take a note, will you!

Q6:

In isosceles triangle ABC, $\overline{AC}=\overline{BC}=10$, $\overline{AB}=12$. $\overline{AD}$ bisects $\angle BAC$ and intersects $\overline{BC}$ at point D.

Find the distance from point D to segment AB.

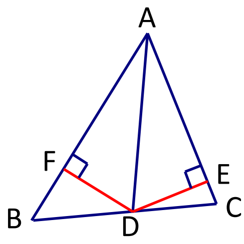


ANSWER

Sol:

Construct $\overline{DF}\perp\overline{AB}$ and $\overline{DE}\perp\overline{AC}$. As shown in Figure 1.

Since $\overline{AD}$ is the angle bisector, $\overline{DF}=\overline{DE}$.



In isosceles triangle ABC, we have lots of perpendicular lines and measurement of segments, it reminds us that we might solve the problem much easier by using the concept of the area of ΔABC.

Figure 1

We construct another perpendicular line

$\overline{CG}\perp\overline{AB}$, and intersects $\overline{AB}$ at point G,

as shown in Figure 2

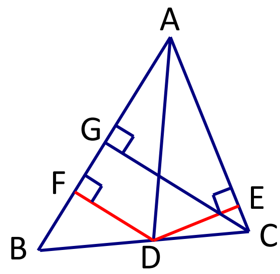


Figure 2

The area of $\Delta ABC=\frac{1}{2}\overline{CG}\cdot\overline{AB}$

How do we get the length of $\overline{CG}$?

In isosceles triangle ABC, $\overline{CG}\perp\overline{AB}$, then $\overline{CG}$ also bisects $\overline{AB}$.

That means: $\overline{AG}=\overline{BG}=\frac{1}{2}\overline{AB}=6$ (Given $\overline{AB}=12$)

($\overline{AC}=\overline{BC}$, then point C must lie on the perpendicular bisector of $\overline{AB}$.)

Let's look at the right triangle ACG,

$$\overline{CG}^2=\overline{AC}^2-\overline{AG}^2$$

$$=10^2-6^2$$

$$=64$$

$$\overline{CG}=8$$

Let’s go back to the equation

$$\text{The area of } \Delta ABC = \frac{1}{2}\overline{CG}\cdot\overline{AB}$$

$$= \frac{1}{2} 8 \cdot 12$$

$$=48$$

On the other hand, we can get the area of ΔABC by adding up the area of ΔABD and the area of ΔACD.

$$\text{The area of } \Delta ABC = \text{ the area of } \Delta ABD + \text{ the area of } \Delta ACD$$

$$48 = \frac{1}{2}\overline{DF}\cdot\overline{AB} + \frac{1}{2}\overline{DE}\cdot\overline{AC}$$

$$48 = \frac{1}{2} x \cdot 12 + \frac{1}{2} x \cdot 10$$

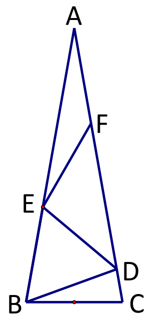
Get $x = \frac{48}{11}$ #

So the distance from point D to segment AB is $\frac{48}{11}$.

Q7:

In isosceles $\triangle ABC$, $\overline{AB}=\overline{AC}$, $\overline{AF}=\overline{FE}=\overline{ED}=\overline{DB}=\overline{BC}$.

Find the measure of ∠A.



ANSWER

Sol:

Ha! We are going to apply lots of triangle exterior angle theorem again.

Let the measure of ∠A is x.

In isosceles triangle AEF, ∠DFE=∠A.+ ∠AEF=2x.

(Triangle exterior angle theorem)

In isosceles triangle DEF, ∠EDF=∠DFE=2x. (The property of isosceles triangles)

In $\triangle$ADE, ∠BED=∠A.+ ∠ADE=X+2x=3X (Triangle exterior angle theorem)

In isosceles triangle BDE, ∠BED=∠DBE=3x (The property of isosceles triangles)

In $\triangle$ADB,∠BDC=∠A.+ ∠ DBE =x+3x=4x (Triangle exterior angle theorem)

In isosceles triangle BCD, ∠C=∠BDC=4x. (The property of isosceles triangles)

And $\triangle$ABC is an isosceles triangle,

∠C=∠ABC=∠ABD.+ ∠CBD (The property of isosceles triangles)

4x=3x+ ∠CBD

∠CBD=x

In isosceles triangle ABC,

∠A.+ ∠ABC+ ∠C=180°

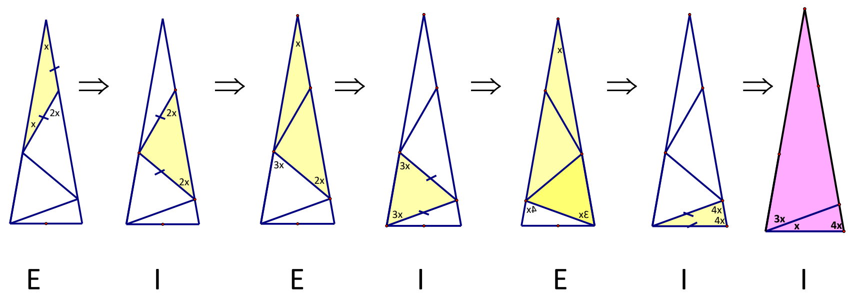
That is:

X+4X+4X=180°

X=20° #

So the measure of ∠A is 20°

FOR YOUR REFERENCE(**E:**stands for 外角定理，**I:** stands for 等腰三角形)



Q8:

The side length of an equilateral triangle is 6.

Find

(1) The length of the height and

(2) The area

of this equilateral triangle.

ANSWER

Sol:

(1) The length of the height of this equilateral triange

$$=\frac{\sqrt{3}}{2}\cdot 6$$

$$=3\sqrt{3}$$

(2) The area of this equilateral triangle

$$=\frac{\sqrt{3}}{4}\cdot 6^2$$

$$=9\sqrt{3}\ \#$$

Reminder:

The side length of an equilateral triangle is a, then

The height of this equilateral triangle is

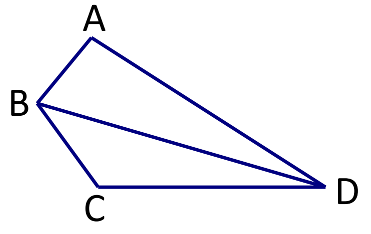
$$\frac{\sqrt{3}}{2}a$$

And the area of this equilateral triangle is

$$\frac{\sqrt{3}}{4}a^2$$

Q9:

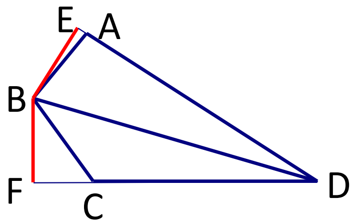
In quadrilateral ABCD, $\overline{BD}$ bisects $\angle ADC$. $\overline{AD}=11$, and $\overline{CD}=7$. If the area of triangle ABD is 33, what is the area of quadrilateral ABCD?



ANSWER

Sol:

(Since it talks about the area, we need more perpendicular lines for sure.)



Construct $\overline{BE} \perp \overrightarrow{DA}$, $\overline{BF} \perp \overrightarrow{DC}$,

$\overline{BE}$ and $\overline{BF}$ intersects $\overrightarrow{DA}$ and $\overrightarrow{DC}$ at point E and point F respectively.

$\overline{BD}$ bisects $\angle ADC$, then $\overline{BE} = \overline{BF}$ (The property of angle bisector)

Let aΔ stand for the area of a triangle.

Then

$$\frac{a\triangle CBD}{a\triangle ABD} = \frac{\frac{1}{2}\overline{BF}\cdot\overline{CD}}{\frac{1}{2}\overline{BE}\cdot\overline{AD}} = \frac{\overline{CD}}{\overline{AD}} = \frac{7}{11}$$

$$a\Delta CBD = \frac{7}{11} a\triangle ABD$$

So the area of quadrilateral ABCD= aΔCBD+ aΔABD

$$= \frac{7}{11} a\triangle ABD + a\Delta ABD$$

$$= \frac{18}{11} a\triangle ABD$$

$$= \frac{18}{11} \cdot 33$$

=54#

Q10:

Two points A and B are on the same side of line L, as shown in Figure 1.

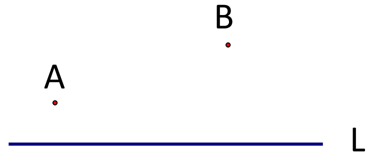


Figure 1

(1) Construct and explain:

Find a point P on line L such that the value of $\overline{PA}+\overline{PB}$ is the least

(Hint: apply the property of perpendicular bisectors.)

(2) In Figure 2, $\overline{AC} \perp \overline{CD}$, $\overline{BD} \perp \overline{CD}$. If $\overline{AC}=2$, $\overline{BD}=4$, and $\overline{CD}=8$.

Find the minimum value of $\overline{PA}+\overline{PB}$ where point P lies on $\overline{CD}$.

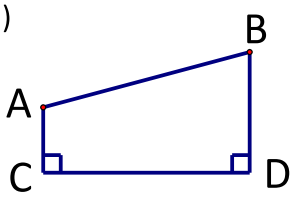


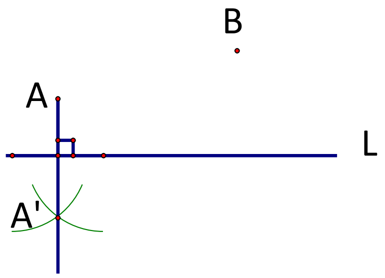
Figure 2

ANSWER

Sol:

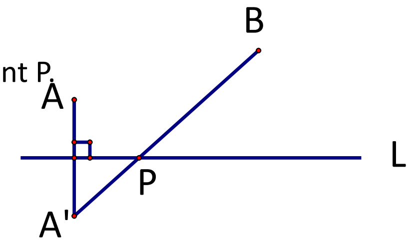
(1) Construction

① Draw a point A’ such that points A and A’ are symmetric to line L.



② Connect $\overline{A'B}$. $\overline{A'B}$ intersects line L at point P.

Then point P is as requested.



Explanation:

Get an arbitrary point Q on line L.

Connect $\overline{QA'}$ and $\overline{QB}$.

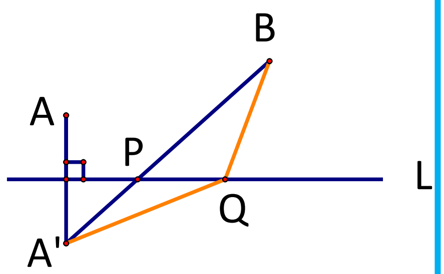
In $\Delta A'BQ$, $\overline{QA'}+\overline{QB}>\overline{A'B}$

(Triangle inequality theorem)

But $\overline{PA}=\overline{PA'}$, $\overline{QA}=\overline{QA'}$

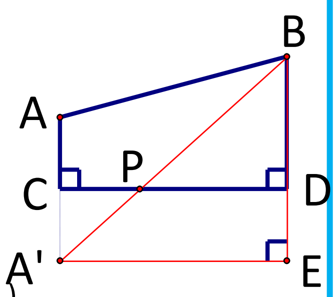
(The property of perpendicular bisectors)

So $\overline{QA}+\overline{QB}=\overline{QA'}+\overline{QB}>\overline{A'B}=\overline{PA'}+\overline{PB}=\overline{PA}+\overline{PB}=\overline{A'B}$



Therefore, for any arbitrary point Q on line L, the statement

$\overline{QA}+\overline{QB}>\overline{PA}+\overline{PB}$ is always true.



(2) Points A’ and A are symmetric to $\overline{CD}$.

Construct a right triangle A’BE as shown in Figure 3.

$\overline{BE}=\overline{BD}+\overline{DE}$ =4+2=6, and $\overline{A'E}=\overline{CD}$ =8.

Then $\overline{PA}+\overline{PB}=\overline{PA'}+\overline{PB}=\overline{A'B}$

$\overline{A'B}^2=\overline{A'E}^2+\overline{BE}^2$ (The Pythagorean theorem,)

$=8^2+6^2$

$=100$

$\overline{A'B}=10$#

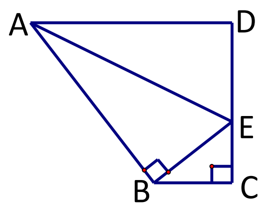
The minimum value of $\overline{PA}+\overline{PB}$ is 10.

QUESTIONS ONLY:

Q1:

In trapezoid ABCD, $\overline{AD} // \overline{BC}$, $\angle C = \angle ABE = 90°$, $\overline{AE}$ is the angle bisector of $\angle BAD$.

If $\overline{CD} = 10$, $\overline{BC} = 2\sqrt{5}$, find the measure of $\overline{BE}$.



Q2:

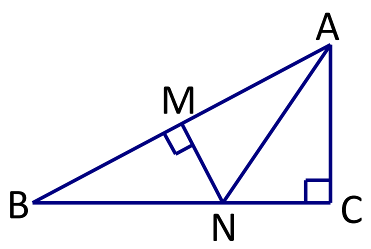
ΔABC is a right triangle, $\angle C = 90°$, $\overline{MN} \perp \overline{AB}$, and $\overline{AN} = \overline{BN}$

If $\overline{AB} = 10$, $\overline{AC} = 6$,

Find (1) The length of $\overline{AM}$

(2) The length of $\overline{AN}$

(3) The area of ΔAMN

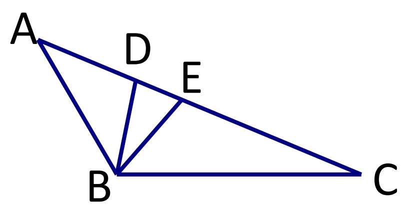


Q3:

In ΔABC, points D and E are on $\overline{AC}$,

$\overline{AB} = \overline{AE}$, $\overline{CB} = \overline{CD}$.

If $\angle ABC = 120°$, find the measure of $\angle DBE$.

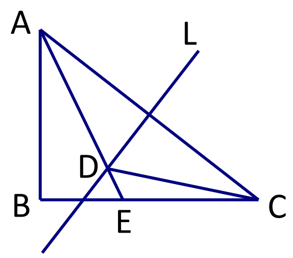


Q4:

In ΔABC, $\angle ABC = 90°$, $\overline{AE}$ bisects $\angle BAC$, and intersects $\overline{BC}$ at point E.

Line L is the perpendicular bisector of $\overline{AC}$, and intersects $\overline{AE}$ at point D.

If $\angle DCE = 24°$, find the measure of $\angle CDE$.

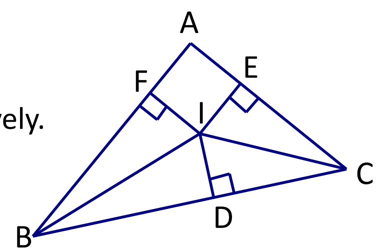


Q5:

Point I is an interior point of ΔABC,

Point P is equidistant from $\overline{AB}$, $\overline{BC}$ $\overline{CA}$ respectively.

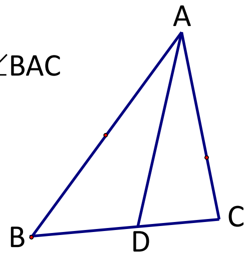
If $\angle A = 70°$, find the measure of $\angle BIC$.



Q6:

In isosceles triangle ABC, $\overline{AC}=\overline{BC}=10$, $\overline{AB}=12$. $\overline{AD}$ bisects $\angle BAC$ and intersects $\overline{BC}$ at point D.

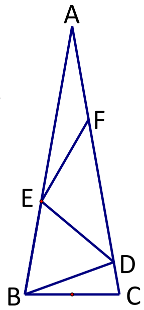
Find the distance from point D to segment AB.



Q7:

In isosceles ΔABC, $\overline{AB}=\overline{AC}$, $\overline{AF}=\overline{FE}=\overline{ED}=\overline{DB}=\overline{BC}$.

Find the measure of $\angle A$.



Q8:

The side length of an equilateral triangle is 6.

Find

(1) The length of the height and

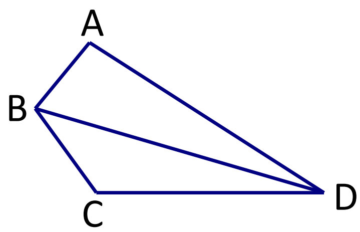
(2) The area

of this equilateral triangle.

Q9:

In quadrilateral ABCD, $\overline{BD}$ bisects $\angle ADC$.

$\overline{AD}=11$, and $\overline{CD}=7$. If the area of triangle ABD is 33, what is the area of quadrilateral ABCD?



Q10:

Two points A and B are on the same side of line L, as shown in Figure 1.

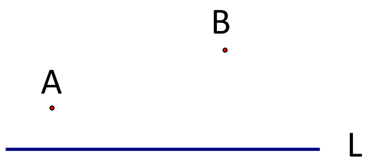


Figure 1

(1) Construct and explain:

Find a point P on line L such that the value of $\overline{PA}+\overline{PB}$ is the least

(Hint: apply the property of perpendicular bisectors.)

(2) In Figure 2, $\overline{AC}\perp\overline{CD}$, $\overline{BD}\perp\overline{CD}$. If $\overline{AC}=2$, $\overline{BD}=4$, and $\overline{CD}=8$.

Find the minimum value of $\overline{PA}+\overline{PB}$ where point P lies on $\overline{CD}$.

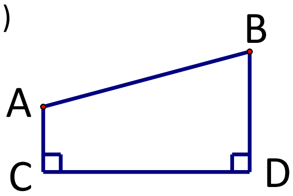


Figure 2

Reference:

教育部國民中學數學 108 課綱
教育部審定國民中學數學科南一、康軒以翰林及第五冊課本

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