雙語教學主題(國中八年級下學期教材)中垂腺與角平分線

Topic: Applications of triangle congruency

Perpendicular bisectors and angle bisectors

Vocabulary

CPCTC stands for corresponding parts of the congruent triangles are congruent		兩全等三角形的對應邊和對應角都對應相等	
Triangle congruence theorem		全等三角形全等性質(定理)	
Isosceles triangle	等腰三角形	two legs	兩腰
base angles	底角	vertex angle	頂角
angle bisector	角平分線	intersect	相交於
reflexive property	反身性(共用邊角)	bisect	平分
perpendicular bisector	垂直平分線	The Pythagorean Theorem	畢氏定理
equilateral triangle	正三角形	quadrilateral	四邊形
take the square root of both sides	等號兩邊開根號	ruler-compass construction	尺規作圖
respectively	按順序地	right triangle	直角三角形
arbitrary point	任意點	intersection point	交點
perimeter	周長	converse theorem	逆定理
perpendicular distance	(點到線的) 垂直距離	equidistant	等距離的
measure	度量、測量		

(我個人很喜歡這個 CPCTC 的表示法，否則要寫很多中文字…)

老師們好，內容儘量呈現在教學時可能用到的英語，供老師們參考。可以自行節錄選取需要的段落或語句使用。也希望老師們建議和指正。祝教學愉快!

In the last class, we introduced five triangle congruence theorems. Hope you have already kept them in mind. Because we are going to discuss many important applications related to triangle congruency, these important applications and properties are often utilized when solving geometric problems. You will find it easier to learn this lesson if you fully understand these theorems.

Are you ready? Let’s go.

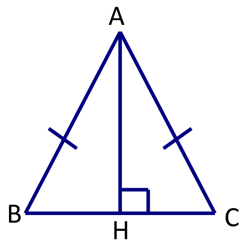
We first look into some classic shapes of triangles like isosceles triangles, equivalent triangles, and right triangles. The properties of these triangles are so often used shortly, please study this lesson with hearts.

Isosceles triangle: a triangle has two equal sides. They are called two legs.

In an isosceles tirangle:

(1) two base angles are congruent

(2) the angle bisector of the vertex angle of an isosceles angle is a perpendicular bisector of the base



Flash prove:

(1) In ΔABC, two legs $\overline{AB}$ and $\overline{AC}$ are congruent as shown on the right.

Construct $\overline{AH} \perp \overline{BC}$ through point A and

$\overline{AH}$ intersects $\overline{BC}$ at point H.

Now in ΔABH and ΔACH,

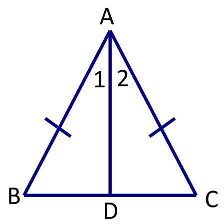
$\overline{AB} = \overline{AC}$ (Given)

$\overline{AH} = \overline{AH}$ (Reflexive property)

$\angle AHB = \angle AHC = 90^\circ$ ($\overline{AH} \perp \overline{BC}$)

We have $\Delta ABH \cong \Delta ACH$ (RHS)

Then $\angle B = \angle C$ (CPCTC)



(2) Construct angle bisector $\overline{AD}$ of $\angle BAC$,

$\overline{AD}$ intersects $\overline{BC}$ at point D.

In ΔABD and ΔACD,

$\overline{AB} = \overline{AC}$ (Given)

$\overline{AD} = \overline{AD}$ (Reflexive property)

$\angle 1 = \angle 2$ (Definition of angle bisector)

$\Delta ABD \cong \Delta ACD$ (SAS)

Then $\overline{BD} = \overline{CD}$ and

$\angle ADB = \angle ADC$ (CPCTC)

However, $\angle ADB + \angle ADC = 180^\circ$ (Points B, D, and C are collinear)

So $\angle ADB = \angle ADC = 90^\circ$

We conclude that $\overline{AD}$ is the perpendicular bisector of $\overline{BC}$.

It shows that when we have a vertex angle bisector in an isosceles triangle, the angle bisector also bisects the base and is perpendicular to the base.

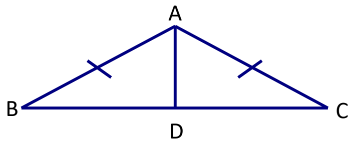
Question:

In an isosceles triangle, if line L bisects the base through vertex A, will line L bisect its vertex angle? Will line L be perpendicular to the base?

Or, in an isosceles triangle, if line L is perpendicular to the base through vertex A, will line L bisect the vertex angle? Will line L bisect the base?

Please discuss these questions with your classmates. Prove it if it's true, or show us a counterexample if you think it's incorrect.

Ex 1:



In ΔABC, $\overline{AB}=\overline{AC}=17$, $\overline{BC}=30$, $\overline{AD}$ bisects $\angle BAC$ and intersects $\overline{BC}$ at point D.

Find (1) The length of $\overline{AD}$

(2) The area of ΔABD

Sol:

(1) ΔABC is an isosceles triangle, $\overline{AD}$ bisects the vertex angle.

So we get $\overline{AD} \perp \overline{BC}$ and $\overline{AD}$ bisects $\overline{BC}$

i.e. $\overline{BD}=\overline{CD}$

so $\overline{BD}=\overline{CD}=\frac{1}{2}\overline{BC}$

$=\frac{1}{2}\cdot 30$

$=15$

In the right triangle ABD,

$\overline{AD}^2=\overline{AB}^2-\overline{BD}^2$ (The Pythagorean Theorem)

$=17^2-15^2$

$=64$

$\Rightarrow \overline{AD}=8$# (The length of a side is positive)

(2) The area of ΔABD

$=\frac{1}{2}\overline{BD}\cdot\overline{AD}$

$=\frac{1}{2}15\cdot 8$

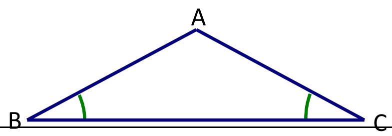
$=60$#

Conversely, if we know two angles in a triangle are equal, is this triangle isosceles?

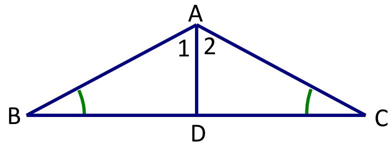
Given $\angle B=\angle C$ in ΔABC is shown on the right.

Is ΔABC isosceles?

Sol:



Construct an angle bisector $\overline{AD}$ of $\angle BAC$,



$\overline{AD}$ intersects $\overline{BC}$ at point D.

We have $\angle 1 = \angle 2$

In $\triangle ABD$ and $\triangle ACD$,

$\angle B = \angle C$ (Given)

$\overline{AD} = \overline{AD}$ (Reflexive property)

$\angle 1 = \angle 2$ (Definition of angle bisector)

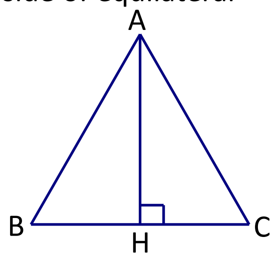
$\triangle ABD \cong \triangle ACD$ (AAS)

Then $\overline{AB} = \overline{AC}$ (CPCTC)

We get that $\triangle ABC$ is isosceles

Let's move on to the perfect shape—equilateral triangles. We will talk about the height and the area of an equilateral triangle here. Please pay attention to these formulas we mention later, you will see the results of equilateral triangles are wildly used when you deal with geometric problems.

In equilateral triangle ABC, $\overline{AH} \perp \overline{BC}$, let the length of the side of equilateral triangle ABC equal a..



Find (1) The length of the height $\overline{AH}$.

(2) The area of the equivalent triangle ABC.

Sol:

(1) From the previous discussion, equilateral triangles are isosceles too. Since $\overline{AH} \perp \overline{BC}$, $\overline{AH}$ also bisects $\overline{BC}$.

Then $\overline{BH} = \overline{CH} = \frac{1}{2}\overline{BC} = \frac{1}{2}a$ $(\overline{AB} = \overline{BC} = \overline{CA} = a)$

In the right triangle ABH,

$\overline{AH}^2 = \overline{AB}^2 - \overline{BH}^2$ (The Pythagorean Theorem)

$= a^2 - \left(\frac{1}{2}a\right)^2$

$= \frac{3}{4}a^2$

So $\overline{AH} = \frac{\sqrt{3}}{2}a$ # (The length of a side is positive)

(2) The area of equilateral triangle ABC

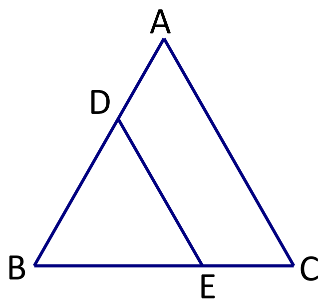
$$=\frac{1}{2}\overline{AH}\cdot\overline{BC}$$

$$=\frac{1}{2}\cdot\frac{\sqrt{3}}{2}a\cdot a$$

$$=\frac{\sqrt{3}}{4}a^2 \#$$

These two formulas from the above discussion will be used a lot when you attend the ninth grade. Review them from time to time.

Ex 2:



ΔABC and ΔDBE are equilateral triangles.

$\overline{AB}=5$, $\overline{DB}=3$. Find the area of quadrilateral ADEC.

Sol:

The area of quadrilateral ADEC

The area of equilateral triangle ABC

-the area of equilateral triangle DEC

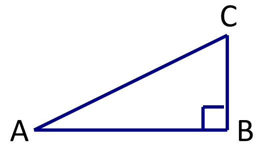
$$=\frac{\sqrt{3}}{4}\cdot 5^2-\frac{\sqrt{3}}{4}3^2$$

$$=4\sqrt{3} \#$$

Remember the Pythagorean theorem we learned in the eighth grade? It says:

In the right triangle ABC, $\angle B=90^\circ$, the relation of the three sides of this triangle is:

$$\overline{AC}^2=\overline{AB}^2+\overline{BC}^2$$



Many of us always think it's ok to say:

In ΔABC, if the lengths of three sides satisfy the relation

$\overline{AC}^2=\overline{AB}^2+\overline{BC}^2$, then ΔABC must be a right triangle.

But is the statement above correct?

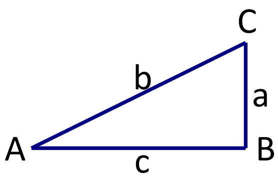
Talk to your classmates, and see if you can find any counterexamples.

Teachers can control their timing and let students have a free discussion.

Let's prove that the converse theorem is also correct.

(下面的證明，我是參考南一版的內容。 如果老師們有更好或是特別的證明，希望老師們可以跟我分享，我趁機多學一點。先謝謝老師們。)

In ΔABC as shown in Figure 1, $\overline{AB}=c$, $\overline{AC}=b$, and $\overline{BC}=a$.



If $b^2=a^2+c^2$, then ΔABC is a right triangle.

Pf:

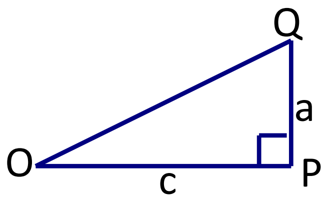
(It's not easy to get something out of nothing!

We need to get the result on angles, but we have only the information on three sides from the given. So we create new information to help the proving.)

Let's construct a right triangle OPQ where $\angle P=90^\circ$,

$\overline{OP}=c$, and $\overline{PQ}=a$.

By the pythagorean theorem, we know that



$$\overline{OQ}^2=\overline{PQ}^2+\overline{OP}^2$$

$$=a^2+c^2$$

And in ΔABC,

$b^2=a^2+c^2$ (Given)

It implies

$\overline{OQ}^2=b^2 \Rightarrow \overline{OQ}=b$ (Take the square root of both sides and the length of a segment is always posive)

Now look back into ΔABC and ΔOPQ,

$\overline{AB}=c=\overline{OP}$

$\overline{BC}=a=\overline{PQ}$ (From the construction we made)

$\overline{AC}=b=\overline{OQ}$ (Previous result)

$\Delta ABC \cong \Delta OPQ$ (SSS)

(So we get what we want.)

Then

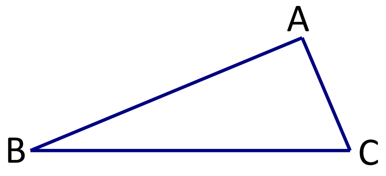
$\angle B=\angle P = 90^\circ$ (CPCTC)

That means

ΔABC is a right triangle. WALA!

We prove that the converse theorem of the Pythagorean theorem is true.

Ex 3:



In $\triangle ABC$, $\overline{AB}=15$, $\overline{AC}=8$, and $\overline{BC}=17$.

Find the area of $\triangle ABC$.

Sol:

Since 8^2+15^2

$=64+225$

$=289$

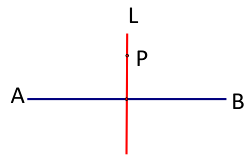
$=17^2$

We know that $\triangle ABC$ is a right triangle and $\angle A=90°$

The area of $\triangle ABC=\frac{1}{2}\cdot 8\cdot 15=60$#

In the ruler-compass construction section, we learned how to construct a perpendicular bisector of a line segment and an angle bisector of an angle. In this class, we are going to explore some very fundamental properties of perpendicular bisector lines and angle bisector lines. They will be intensively used when we deal with geometrical problems. Please concentrate, ok?

Perpendicular bisector :



Line L is the perpendicular bisector of segment AB.

An arbitrary point P is on line L. Then $\overline{PA}=\overline{PB}$

Pf:

Mark the given information in Figure 1.

Point M is the intersection point of line L and $\overline{AB}$

Then in Figure 1,

$\overline{MA}=\overline{MB}$ and $\angle PMA=\angle PMB=90°$

(Definition of perpendicular bisector)

In $\triangle PMA$ and $\triangle PMB$,

$\overline{MA}=\overline{MB}$

$\angle PMA=\angle PMB=90°$ (Given)

$\overline{PM}=\overline{PM}$ (Reflexive property)

Then

$\triangle PMA\cong\triangle PMB$ (SAS)

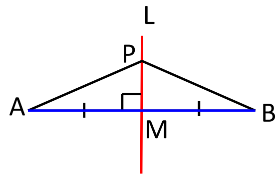


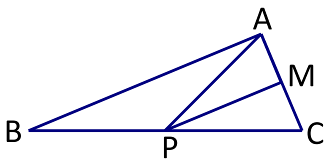
Figure 1

We get

$\overline{PA}=\overline{PB}$ (CPCTC)

Let's see how we apply this property to the problems.

Ex 4:



In $\triangle ABC$, $\angle BAC=90°$, $\overline{PM}$ is the perpendicular bisector of $\overline{AC}$. $\overline{PM}$ intersects $\overline{AC}$ and $\overline{BC}$ at point M and point P respectively. $\overline{AB}=8$ and $\overline{AC}=6$. Find the perimeter of $\triangle ABP$.

Sol:

$\angle BAC=90°$, $\triangle ABC$ is a right triangle, by the Pythagorean theorem,

we get $\overline{BC}^2=\overline{AB}^2+\overline{AC}^2$

$=8^2+6^2$

$=100$

$\overline{BC}=10$ (Take the square root of both sides)

$\overline{PM}$ is the perpendicular bisector of $\overline{AC}$,

then $\overline{PA}=\overline{PC}$ (Property of perpendicular bisector)

The perimeter of $\triangle ABP=\overline{AB}+\overline{BP}+\overline{PA}$

$=\overline{AB}+\overline{BP}+\overline{PC}$ ($\overline{PA}=\overline{PC}$)

$=\overline{AB}+\overline{BC}$ ($\overline{BP}+\overline{PC}=\overline{BC}$)

$=8+10$

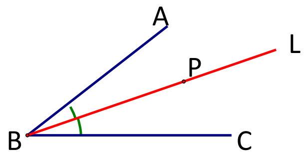
$=18$#

Let's move on to the property of an angle bisector.

Angle bisector :

Line L is the angle bisector of angle ABC through point B.

An arbitrary point P is on line L. Then point P is equidistant from $\overline{BA}$ and $\overline{BC}$.



Pf:

Mark the given information in Figure 1. Point P is an arbitrary point on line L,

$\overline{PM}\perp\overline{BA}$ and $\overline{PN}\perp\overline{BC}$

(Whenever we talk about the distance from a point to a line,, we always refer to the "perpendicular distance.")

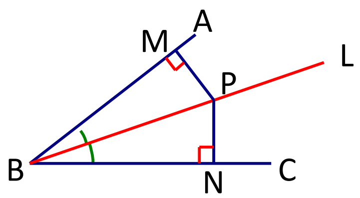


Figure 1

In $\triangle BMP$ and $\triangle BNP$,

$\angle BMP=\angle BNP=90°$ ($\overline{PM}\perp\overline{BA}$, $\overline{PN}\perp\overline{BC}$)

$\angle PBA=\angle PBC$ (Definition of angle bisector)

$\overline{PB}=\overline{PB}$ (Reflexive property)

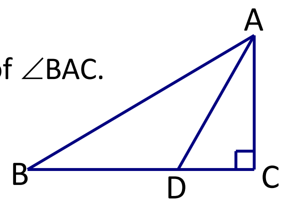
Then	$\Delta BMP \cong \Delta BNP$	(AAS)
	$\overline{PM} = \overline{PN}$	(CPCTC)

Let's look at an example.

Ex 5:

In right triangle ABC, $\angle C = 90°$, $\overline{AD}$ is the angle bisector of $\angle BAC$.

If $\overline{AB} = 15$, $\overline{CD} = 4$. Find the area of ΔABD.



Sol:

Since $\overline{AD}$ is the angle bisector of $\angle BAC$, if we construct $\overline{DE} \perp \overline{AB}$, as shown in Figure 1.

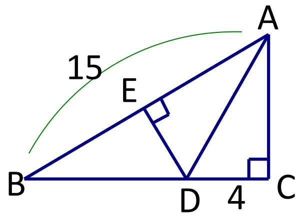


Figure 1

Then $\overline{DE} = \overline{CD} = 4$ (Property of angle bisector)

In this case,

The area of $\Delta ABD = \frac{1}{2}\overline{DE} \cdot \overline{AB}$

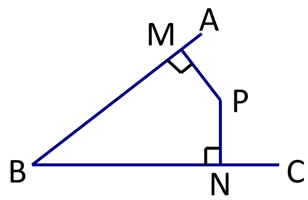
$= \frac{1}{2} 4 \cdot 15$

$= 30_{\#}$

How about the converse statement of the angle bisector property? We'll discuss it as follows. You will easily understand the converse satement is true as well.

In the figure on the right, point P is inside $\angle ABC$,.

$\overline{PM} \perp \overline{BA}$ and $\overline{PN} \perp \overline{BC}$. If $\overline{PM} = \overline{PN}$, then point P must lie on the angle bisector of $\angle ABC$.



Pf:

Connect $\overline{PB}$ as shown in Figure 1.

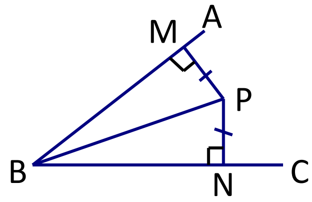


Figure 1

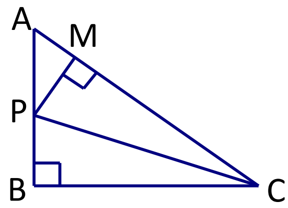
In ΔBMP and ΔBNP,

	$\angle BMP = \angle BNP = 90°$	($\overline{PM} \perp \overline{BA}$, $\overline{PN} \perp \overline{BC}$)
	$\overline{PM} = \overline{PN}$	(Given)
	$\overline{PB} = \overline{PB}$	(Reflexive property)
Then	$\Delta BMP \cong \Delta BNP$	(RHS)
	$\angle PBA = \angle PBC$	(CPCTC)

This means $\overline{PB}$ is the angle bisecor of $\angle ABC$, of couse point P lies on the angle bisector $\overline{PB}$.

Here is an example of it.

Ex 6:



In the right triangle ABC, ∠B=∠CMP=90°. $\overline{PB}=\overline{PM}$.

If ∠A=50°, find the measure of ∠APC.

Sol:

Since ∠B=∠CMP=90°. $\overline{PB}=\overline{PM}$, we know that

point P is on the angle bisector $\overline{PC}$ of ∠ACB, that is. ∠ACP=∠BCP.

In △ABC,

∠ACB=180°-∠A-∠B

=180°-50°-90°

=40°

So ∠ACP=$\frac{1}{2}$∠ACB=20°

We get ∠APC=180°-∠A-∠ACP

=180°-50°-20°

=110° #

The properties of perpendicular bisectors and angle bisectors are so useful when you solve geometric problems. Please learn them well.

We'll do more practice next time.

Reference:

教育部國民中學數學 108 課綱

教育部審定國民中學數學科南一、康軒以翰林及第五冊課本