乘法公式
Short Multiplication Formulas

Class:__________ Name: __________

1. Distributive property

We learned the distributive property in elementary school and it was widely used in the seventh grade. Let's review the property first.

$a \cdot (b+c)=ab+ac$

We use this property to simplify the expression with a variable, such as $3(2x+1)=6x+3$. However, the property is also useful when we do some basic calculations. For example, when calculating 102×37, we can write 102 as $100+2$, and then apply the distributive property.

$102\times37=(100+2)\times37=100\times37+2\times37=3774$

If we also want to write 37 as $30+7$, can we still use this method? The answer is yes, and we need to use the distributive property twice. We can use the same process as above, but we see $30+7$ as a whole.

$102\times37=(100+2)\times\boxed{(30+7)}=100\times\boxed{(30+7)}+2\times\boxed{(30+7)}$

Next, it is easy to see that we can apply for the property again.

$100\times(30+7)+2\times(30+7)=100\times30+100\times7+2\times30+2\times7=3000+700+60+14=3774$

Therefore, the product of 102×37 is 3774.

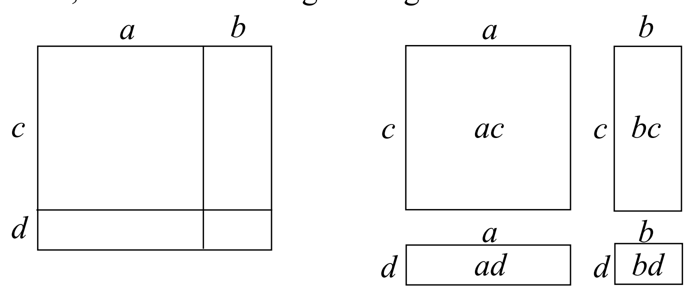
Generally, we have the following conclusion.

$(a+b)(c+d)=ac+ad+bc+bd$

First, we prove the property in the above process. We just turn the numbers into symbols.

$(a+b)\boxed{(c+d)}=a(c+d)+b(c+d)=ac+ad+bc+bd$

We can also use the graph to prove the property above. Let's look at the rectangle with length $(a+b)$ and width $(c+d)$. What is the area of this rectangle? We can divide this rectangle into four small rectangles. It is easy to get the area of these rectangles is ac, ad, bc, and bd respectively. Therefore, the area of the big rectangle is $ac+ad+bc+bd$.



Example 1

Calculate 998×2003

[Solution]

We can apply the distributive property twice as the previous process. 998 can be written as $(1000-2)$, and 2003 can be written as $(2000+3)$.

998×2003

$=(1000-2)(2000+3)$

$=1000 \times (2000+3)-2 \times (2000+3)$

$=1000 \times 2000+1000 \times 3-2 \times 2000-2 \times 3$

$=2000000+3000-4000-6$

$=1998994$

If we are familiar with the process, we can also apply the property faster. The following four arrows simplify the process of multiplication.

998×2003

$=(1000-2)(2000+3)$

$=1000 \times 2000+1000 \times 3-2 \times 2000-2 \times 3$

$=2000000+3000-4000-6$

$=1998994$

Exercise 1

Calculate 508×1999

2. Square of a sum(和的平方)

Next, we introduce a tip to calculate the square of a sum. Take 205^2 for example. 205 can be written as $(200+5)$, and the square of a number means the numbers multiplied by itself twice. Therefore, $205^2=(200+5)(200+5)$. Then, we can apply the distributive property which we just mentioned.

$(200+5)(200+5)$

$=200 \times 200+5 \times 200+200 \times 5+5 \times 5$

$=40000+1000+1000+25$

$=42025$

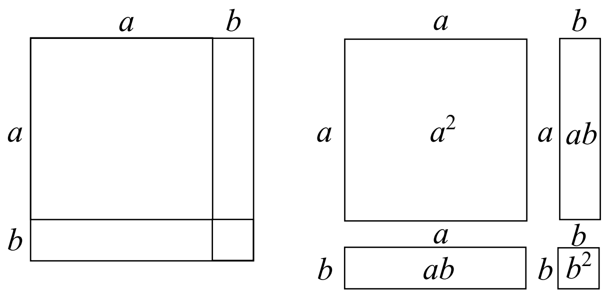
If we change the numbers into symbols, what can we get? Let 200 be a, and 5 be b, and then

$$(a+b)^2=(a+b)(a+b)=a^2+ba+ab+b^2=a^2+ab+ab+b^2=a^2+2ab+b^2$$

[Square of a sum]

$$(a+b)^2=a^2+2ab+b^2$$

We can also use the graphs to prove this. In the following figure, suppose there is a square with the length $(a+b)$, and then the area of the square is $(a+b)^2$. Dividing the squares into four pieces, we have two squares with the area of a^2 and b^2 respectively, and two rectangles with the area of $2ab$. Therefore, the area of the original square $(a+b)^2$ is equal to $a^2+2ab+b^2$.



The formula is important. We had better keep it in mind and directly apply it if we need to. Here is an example of applying the formula.

Example 2

Calculate 40.3^2

[Solution]

First, we can write 40.3 as $(40+0.3)$. We know that $(a+b)^2=a^2+2ab+b^2$. Let a be 40 and b be 0.3.

$$(40+0.3)^2=40^2+2\times40\times0.3+0.3^2=1600+24+0.09=1624.09$$

$$(a+b)^2=a^2+2ab+b^2$$

Exercise 2

Calculate 100.5^2

We can also apply the formula from right to left. Let's see the next example.

Example 3

Calculate $596^2+2\times596\times4+4^2$

[Solution]

$$596^2+2\times596\times4+4^2=(596+4)^2=600^2=360000$$

$$a^2+2ab+b^2=(a+b)^2$$

Exercise 3

Calculate $27^2 + 2 \times 27 \times 3 + 3^2$

3. Square of a difference(差的平方)

How about calculating 999^2? 999 can be written as $(1000-1)$, and 999^2 equals 999×999. Therefore, $999^2 = (1000-1)(1000-1)$. We can apply the distributive property again.

$$(1000-1)(1000-1)$$

$$= 1000 \times 1000 - 1000 \times 1 - 1 \times 1000 + 1 \times 1$$

$$= 1000000 - 1000 - 1000 + 1$$

$$= 998001$$

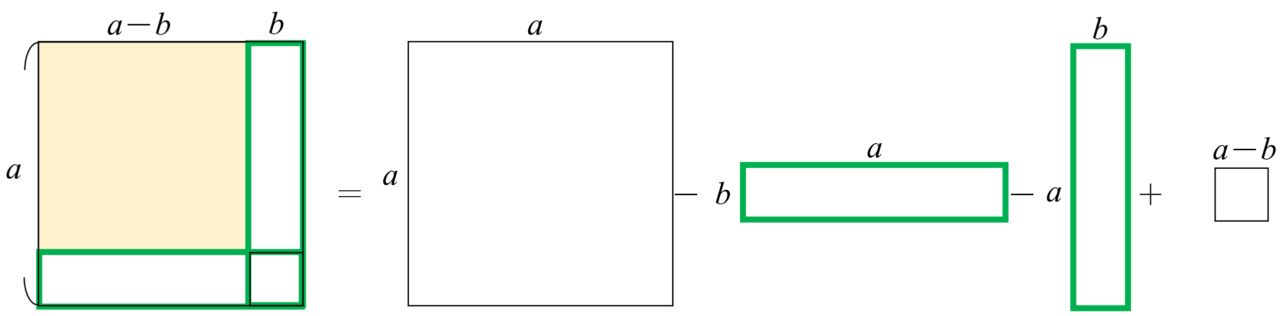
In general, we let 1000 be a and 1 be b, and then

$$(a-b)^2 = (a-b)(a-b) = a^2 - ab - ba + b^2 = a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$$

[Square of a difference]

$$(a-b)^2 = a^2 - 2ab + b^2$$

We can also use the graphs to prove this. To prove the formula, we have to find a square with side length $(a-b)$. To calculate the area, start with a square of side length a and subtract the excess parts. We can subtract two rectangles whose length is a and width is b. Then, we find a small square that has been subtracted twice, so we have to add back the area, which is b^2. Therefore, the area of the square with side length $(a-b)$ equals $a^2 - ab - ab + b^2 = a^2 - 2ab + b^2$. We get the same result as above.



Here is an example to show how to apply the square of a difference formula.

Example 4

Calculate $\left(13\frac{5}{7}\right)^2$

[Solution]

First, write $13\frac{5}{7}$ as $(14-\frac{2}{7})$. We know that $(a-b)^2=a^2-2ab+b^2$. Let a be 14 and b be $\frac{2}{7}$.

$$(14-\frac{2}{7})^2=14^2-2\times14\times\frac{2}{7}+(\frac{2}{7})^2=196-8+\frac{4}{49}=188\frac{4}{49}$$

$$(a+b)^2=a^2-2ab+b^2$$

Exercise 4

Calculate $\left(19\frac{3}{4}\right)^2$

We can also use the formula from right to left.

Example 5

Calculate $107^2-2\times107\times7+49$

[Solution]

We know that 49 is 7 squared. Therefore,

$$107^2-2\times107\times7+49=107^2-2\times107\times7+7^2=(107-7)^2=100^2=10000$$

$$a^2-2ab+b^2=(a-b)^2$$

Exercise 5

Calculate $304^2-2\times300\times4+16$

4. Difference of squares(平方差)

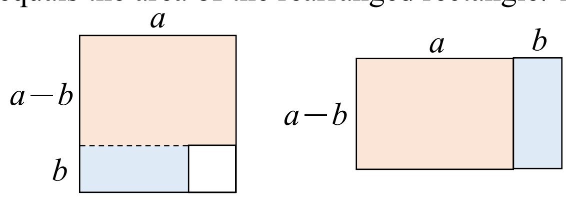
We just learned the formula of $(a+b)^2$, which is $(a+b)$ times $(a+b)$, and the formula of $(a-b)^2$, which is $(a-b)$ times $(a-b)$. How about $(a+b)$ times $(a-b)$? Similarly, we can use the distributive formula to expand the expression.

$$(a+b)(a-b)=a^2-ab+ba-b^2=a^2-ab+ab-b^2=a^2-b^2$$

[Difference of squares]

$$(a+b)(a-b)=a^2-b^2$$

We can also use the graphs to prove this. The expression a^2-b^2 represents the area of a square with side length a minus a square with side length b. We can cut the figure into two different rectangles as follows and then rearrange them. By doing so, we end up with a new rectangle with length $(a+b)$ and width $(a-b)$. Therefore, the difference between the areas of these two squares equals the area of the rearranged rectangle. That is, $a^2-b^2=(a+b)(a-b)$.



Let's see how to apply the formula.

Example 6

Calculate 403×397

[Solution]

Both of these two numbers have a special relationship with 400. 403 can be written as $400+3$, and 397 can write as $400-3$. Then, let a be 400 and b be 3. We have

$$(400+3)(400-3)=400^2-3^2=160000-9=159991$$

$$(a+b)(a-b)=a^2-b^2$$

Exercise 6

Calculate 996×1004

We can also apply the formula from right to left.

Example 7

Calculate 65^2-35^2

[Solution]

We can find that 65 plus 35 equals 100, which can make the multiplication easier. Then, applying the difference of squares formula, we have

$$65^2-35^2=(65+35)(65-35)=100\times 30=3000$$

$$a^2-b^2=(a+b)(a-b)$$

Exercise 7

Calculate 666^2-334^2

一、設計理念：

1. 國外在計算$(a+b)(c+d)$時，會使用口訣 FOIL，foil 本身為錫箔紙，而在這邊指的是 First、Outer、Inner、Last 四個字的縮寫，也就是乘兩式的前項、外項、內項、後項相加，所以得到 $ac+ad+bc+bd$。
2. 乘法公式的英文可用 short multiplication formulas 或 polynomial identities，但由於臺灣課本是先教乘法公式再教多項式，所以本文採用 short multiplication formulas。
3. 國外有教材會將「和的平方」和「差的平方」合稱為 square of binomial pattern。
4. 國外有教材會將「平方差」稱為 sum and difference pattern。

二、英文詞彙：

中文	英文
分配律	distributive property
和的平方	square of a sum
差的平方	square of a difference
平方差	difference of squares
長方形	rectangle
正方形	square

三、教學參考範例：

<table>
<tr>
<td rowspan="4">1
【分配律】
Distributive property</td>
<td>Example 1
Calculate 998×2003
[Solution]
We can apply the distributive property twice as the previous process. 998 can be written as (1000－2), and 2003 can be written as (2000＋3).
998×2003
＝(1000－2)(2000＋3)
＝1000×(2000＋3)－2×(2000＋3)
＝1000×2000＋1000×3－2×2000－2×3
＝2000000＋3000－4000－6
＝1998994</td>
</tr>
<tr>
<td>We already learned how to multiply large numbers like 998×2003 in elementary school. However, the numbers are tricky to calculate directly. To make things easier, we can use the distributive property to break the problem into smaller parts.
To apply the property, we have to take apart the numbers. There are different ways to do this. For example, we can split 998 into 997 plus 1, or 900 plus 98. However, it is usually easier to work with round numbers like 1000 because multiplying by numbers with lots of zeros simplifies things. Therefore, we can write 998 as 1000 minus 2. Similarly, we can write 2003 as 2000 plus 3. Then, we have rewritten the problem as 1000 minus 2 times 2000 plus 3.
Next, we can apply the method we just learned. Both 1000 and 2 should multiply 2000 plus 3. By distributive property, we have 1000 times 2000 plus 3 minus 2 times 2000 plus 3. Then, we can apply the property again. 1000 times 2000 plus 3 equals 1000 times 2000 plus 1000 times 3, and 2 times 2000 plus 3 equals 2 times 2000 plus 2 times 3. We can quickly find each product and then add them together. Finally, we have 2,000,000 plus 3000 minus 4000 minus 6, which equals 1,998,994.</td>
</tr>
<tr>
<td>If we are familiar with the process, we can also apply the property faster. The following four arrows simplify the process of multiplication.
998×2003
＝(1000－2)(2000＋3)
＝1000×2000＋1000×3－2×2000－2×3
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</tr>
<tr>
<td>We can also use the conclusion that both parts of one number need to multiply both parts of the other number. Therefore, 1000 should multiply both 2000 and 3, and －2 also should multiply 2000 and 3. Then, calculate these four products, and add them together. Finally, we get the same answer.</td>
</tr>
</table>

2 【和的平方】 Square of a sum	Example 2 Calculate 40.3^2 [Solution] First, we can write 40.3 as $(40+0.3)$. We know that $(a+b)^2=a^2+2ab+b^2$. Let a be 40 and b be 0.3. $(40+0.3)^2=40^2+2\times 40\times 0.3+0.3^2=1600+24+0.09=1624.09$ $(a+b)^2=a^2+2\ a\ b+b^2$
	Let's see how to use the square of a sum formula to calculate 40.3 squared. 40.3 is close to 40, so we can write it as 40 plus 0.3. Then, we use the square of a sum formula, which is the square of the sum of a and b equals a squared plus 2 times a times b plus b squared. Let a be 40, and b be 0.3, then we have 40.3 squared equals 40 squared plus 2 times 40 times 0.3 plus 0.3 squared. Therefore, it equals 1600 plus 24 plus 0.09, which equals 1624.09.
	Example 3 Calculate $596^2+2\times 596\times 4+4^2$ [Solution] $596^2+2\times 596\times 4+4^2=(596+4)^2=600^2=360000$ $a^2+2\ a\ b+b^2=(a+b)^2$
	We can directly calculate 596 squared, 2 times 596 times 4, and 4 squared. Then we add these three values together. However, this method is a bit time-consuming, so we want to find out whether there is a smarter way to solve this problem. Observing the expression, we can recognize that this expression matches the form of the square of a sum formula. The first term is a square number, the last term is also a square number, and the middle term is two times two numbers. Comparing the two expressions, we let a be 596 and b be 4. Then, the expression is the same as the square of 596 plus 4. Therefore, it equals 600 squared, which equals 360,000.
3 【平方差】 Difference of squares	Example 6 Calculate 403×397 [Solution] Both of these two numbers have a special relationship with 400. 403 can be written as $400+3$, and 397 can write as $400-3$. Then, let a be 400 and b be 3. We have $(400+3)(400-3)=400^2-3^2=160000-9=159991$ $(a+b)(a-b)=a^2-b^2$
	It's not too difficult to calculate the value of the product directly, but there is a faster way to solve it by recognizing a special relationship between 403 and 397. Notice that 403 and 397 are both close to 400. See 400 as the center, 403 is 400 plus 3, and 397 is 400 minus 3. Then we can apply the difference of squares formula. It equals 400 squared minus 3 squared, which equals 159,991.

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