多項式的乘除
Multiplying and Dividing Polynomials

Class:__________ Name: __________

1. Multiplying a polynomial by a polynomial

Let's start with multiplying a monomial by another monomial. To multiply a monomial by another monomial, we can use the commutative property, associative property, and laws of exponent. In other words, when multiplying variables and numbers, we can rearrange the order as needed. When multiplying several x's together, we can simplify them by using exponential notation.

(1) $5 \cdot 4x^3 = (5 \cdot 4) \cdot x^3 = 20x^3$ (associative property)

(2) $9x \cdot (-2x) = 9 \cdot (-2) \cdot x \cdot x = -18x^2$ (commutative property)

(3) $7x^4 \cdot x^5 = 7 \cdot (x^4 \cdot x^5) = 7x^9$ (laws of exponent)

To calculate it faster, we can multiply the coefficients and the variables separately when a monomial is multiplied by another monomial.

Example 1

Calculate $(-3x^2) \cdot (-6x)$

[Solution]

$(-3x^2) \cdot (-6x)$

$= (-3) \cdot (x^2) \cdot (-6) \cdot (x)$ (associative property)

$= (-3) \cdot (-6) \cdot (x^2) \cdot (x)$ (commutative property)

$= 18x^3$

Exercise 1

Calculate $9x^3 \cdot (-x^3)$

To calculate the product of a monomial and a polynomial, you can use the distributive property.

Example 2

Calculate $3x^2 \cdot (2x+7)$

[Solution]

$3x^2 \cdot (2x+7)$

$= 3x^2 \cdot 2x + 3x^2 \cdot 7$

$= 6x^3 + 21x^2$

Exercise 2

Calculate $(-5x^2+2x)\cdot x$

Next, we are going to introduce how to multiply a polynomial by a polynomial. When multiplying polynomials, use the distributive property to expand the expression and then combine like terms. Similar to adding or subtracting polynomials, we can use the vertical method or the horizontal method to multiply polynomials.

Example 3

Calculate $(2x+7)(4x-3)$

[Method 1] Multiplying polynomials horizontally

$(2x+7)(4x-3)$

$=8x^2-6x+28x-21$ (using distributive property twice)

$=8x^2+22x-21$ (combine like terms)

[Method 2] Multiplying polynomials vertically

Multiplying polynomials vertically is similar to multiplying integers vertically.

$$\begin{array}{r} 2x+7 \\ \times)\ 4x-3 \\ \hline -6x-21 \\ 8x^2+28x\quad\quad \\ \hline 8x^2+22x-21 \end{array}$$

Exercise 3

Calculate $(2x+7)(4x-3)$

Example 4

Calculate $(6x^2-7)(-1+4x)$

[Method 1] Multiplying polynomials horizontally

$(6x^2-7)(-1+4x)$

$=-6x^2+24x^3+7-28x$ (using distributive law twice)

$=24x^3-6x^2-28x+7$ (arrange the polynomial in descending form)

[Method 2] Multiplying polynomials vertically

When using the vertical method to multiply polynomials, we can arrange the terms in descending order before calculation. If a term in one of the polynomials is missing, fill in any missing terms with 0.

$$
\begin{array}{rrrrr}
 & & 6x^2 & +0x & -7 \\
\times) & & & 4x & -1 \\
\hline
 & & -6x^2 & +\ 0x & +7 \\
 & 24x^3 & +0x^2 & -28x & \\
\hline
 & 24x^3 & -6x^2 & -28x & +7
\end{array}
$$

Exercise 4

(1) Calculate $(-5x^2+4)(-x+9)$

(2) Calculate $(-7x+3x^2-2)(4-5x)$

When multiplying polynomials, if the polynomial are expressed as short multiplication formulas, we can expand them right away.

Example 5

(1) Calculate $(5x+2)^2$

(2) Calculate $(4x-9)^2$

(3) Calculate $(3x+8)(3x-8)$

[Solution]

(1) $(a+b)^2=a^2+2\cdot a\cdot b+b^2$

$$\downarrow$$

$(5x+2)^2=(5x)^2+2\cdot(5x)\cdot 2+2^2=25x^2+20x+4$

(2) $(a-b)^2=a^2-2\cdot a\cdot b+b^2$

$$\downarrow$$

$(4x-9)^2=(4x)^2-2\cdot(4x)\cdot 9+9^2=16x^2-72x+81$

(3) $(a+b)(a-b)=a^2-b^2$

$$\downarrow$$

$(3x+8)(3x-8)=(3x)^2-8^2=9x^2-64$

Exercise 5

(1) Calculate $(2x+5)^2$

(2) Calculate $(7x-1)^2$

(3) Calculate $(6x+7)(6x-7)$

2. Dividing a polynomial by a polynomial

When dividing integers, there are two ways to calculate it. For example, when we divide 35 by 7, we can use the vertical method or write division as a fraction and simplify it

[Method 1] Vertical method

$$\begin{array}{r} 5 \\ 7\overline{)\,35} \\ 35 \\ \hline 0 \end{array}$$

[Method 2] Write division as a fraction

$$35\div 7=\frac{\cancel{35}^{5}}{\cancel{7}_{1}}=5$$

We can also use this method to divide polynomials. Let's take a look at an example.

Example 6

Calculate $24x^2\div 4x=$?

[Method 1] Vertical method

$$\begin{array}{r} 6x \\ 4x\overline{)\,24x^2} \\ 24x^2 \\ \hline 0 \end{array}$$

[Method 2] Write division as a fraction

$$24x^2\div 4x=\frac{{}^{6}\cancel{24}x^{\cancel{2}}}{{}_{1}\cancel{4x}}=6x$$

Exercise 6

Calculate $(-27x^2)\div 3x=$?

In general, the vertical method is more useful when dividing polynomials. We use the vertical method to calculate $(6x^2+9x)\div 3x$. The method is similar to what we do when dividing integers.

First, what do we multiply $3x$ by to get $6x^2$? The answer is $2x$.

$$
\begin{array}{r|l}
 & 2x \\
\cline{2-2}
3x & 6x^2+9x \\
 & 6x^2 \\
\cline{2-2}
 & \qquad\ \ 9x
\end{array}
$$

Next, what do we multiply $3x$ by to get $9x$? The answer is 3.

$$
\begin{array}{r|l}
 & 2x+\ 3 \\
\cline{2-2}
3x & 6x^2+9x \\
 & 6x^2 \\
\cline{2-2}
 & \qquad\ \ 9x \\
 & \qquad\ \ 9x \\
\cline{2-2}
 & \qquad\quad\ 0
\end{array}
$$

Therefore, we get $(6x^2+9x)\div 3x=2x+3$

Exercise 7

Calculate $(10x^2-8x)\div 2x=$?

When dividing polynomials, the dividend, divisor, quotient, and remainder are all polynomials. The process starts by arranging both the dividend and divisor in descending order and then performing the division.

In the division of positive integers, the operation is complete when the remainder is less than the divisor or when the remainder is zero. Similarly, in the division of polynomials, the operation is complete when the degree of the remainder is less than the degree of the divisor or when the remainder is zero.

Example 8

Calculate $(5x^2-2x-6)\div(x+3)=$?

$$
\begin{array}{r|ll}
 & 5x-17 & \rightarrow 5x^2\div x=5x;\quad \rightarrow (-17x)\div x=-17 \\
\cline{2-2}
x+3 & 5x^2-\ 2x-\ 6 & \\
 & 5x^2+15x & \rightarrow (x+3)\times 5x=5x^2+15x \\
\cline{2-2}
 & \quad\ -17x-\ 6 & \\
 & \quad\ -17x-51 & \rightarrow (x+3)\times(-17)=-17x-51 \\
\cline{2-2}
 & \qquad\qquad 45 &
\end{array}
$$

Therefore, we get $(5x^2-2x-6)\div(x+3)=5x-17\cdots 45$

Example 8

Calculate $(4x^2-14x+1)\div(2x-5)=$?

3. Division Algorithm(除法原理)

If we divide 46 apples evenly among 6 people, each person can get 7 apples, with 4 apples remaining. We can express the process as follows:

$$46\div 6=7\cdots 4$$

Can we use this result to infer how many apples we have at the beginning? We know that each person gets 7 apples, so 6 people in total have 7x6 apples. Adding the 4 apples that are not distributed, there are 7x6＋4 apples in total. In other words, the original division process can be rewritten as an equation:

$$46 = 7 \times 6 + 4$$

Dividend (被除數)　Divisor (除數)　Quotient (商數)　Remainder (餘數)

This is what we learned in elementary school, which is called the division algorithm:

Dividend ＝ Divisor × Quotient ＋ Remainder

The result can also apply to polynomials.

[Division Algorithm for Polynomial]
Let F and G be polynomials and the F÷G＝Q⋯R, then F＝GxQ＋R.

For example, we know that $(4x^2+6x-3)\div(2x+5)=(2x-2)\cdots 7$, then

$$(4x^2+6x-3)=(2x+5)\times(2x-2)+7$$

Dividend ＝ Divisor × Quotient ＋ Remainder

Example 9

A polynomial A is divided by $x+7$ with a quotient of $2x-3$ and a remainder of 8. Find the polynomial A.

[Solution]

We know that Dividend ＝ Divisor × Quotient ＋ Remainder.

Therefore, polynomial $A=(x+7)(2x-3)+8$

$=2x^2+11x-21+8$

$=2x^2+11x-13$

Exercise 9

A polynomial A is divided by $3x-2$ with a quotient of $4x+3$ and a remainder of -12. Find the polynomial A.

一、設計理念：

1. 國外在計算$(a+b)(c+d)$時，會使用口訣 FOIL，foil 本身為錫箔紙，而在這邊指的是 First、Outer、Inner、Last 四個字的縮寫，也就是乘兩式的前項、外項、內項、後項相加，所以得到 $ac+ad+bc+bd$。
2. 國外做多項式的四則運算時常搭配兩個以上的變數。
3. 乘法公式的英文可用 short multiplication formulas 或 polynomial identities，但由於臺灣課本是先教乘法公式再教多項式，所以本文採用 short multiplication formulas。
4. 國外並沒有特別區分被除數和被除式，均是使用 dividend 一詞。同理，除數與除式均使用 divisor，商數與商式均使用 quotient，餘數與餘式均使用 remainder。

二、英文詞彙：

中文	英文
多項式	polynomial
項	term
係數	coefficient
升冪排列	ascending order
降冪排列	descending order
同類項	like terms
x^2	x squared / x to the power of two
x^3	x cubed / x to the power of three
橫式乘法	multiply horizontally
直式乘法	multiply vertically
乘法公式	short multiplication formulas
被除式/數	dividend
除式/數	divisor
商式/數	quotient
餘式/數	remainder
除法原理	division algorithm

三、教學參考範例：

<table>
<tr>
<td rowspan="4">1
【多項式的乘法】
Multiplying polynomials</td>
<td>Example 3
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<tr>
<td>When multiplying polynomials, we also have two different ways. We use the horizontal method first. In the previous section, we have learned how to use distributive property. Let’s apply for the property. $2x$ times $4x$ and -3 respectively, and then 7 times $4x$ and -3 respectively. Therefore, we have $8x^2-6x+28x-21$. Next, because $-6x$ and $28x$ are like terms, we can combine them. The final answer is $8x^2+22x-21$.</td>
</tr>
<tr>
<td>[Method 2] Multiplying polynomials vertically
Multiplying polynomials vertically is similar to multiplying integers vertically.
$\begin{array}{r} 2x+7 \\ \times)\ 4x-3 \\ \hline -6x-21 \\ 8x^2+28x \\ \hline 8x^2+22x-21 \end{array}$</td>
</tr>
<tr>
<td>We can also use the vertical method to multiply polynomials. It is similar to multiplying integers vertically. Write down the polynomials in vertical format, and then multiply each term in the first polynomial by each term in the second polynomial. Starting with the rightmost term, we multiply the first polynomial by -3. 7 times -3 equals -21, and $2x$ times -3 equals $-6x$. Next, at the next row, shift one space to the left, and multiply the first polynomial by $4x$. 7 times $4x$ equals $28x$, and $2x$ times $4x$ equals $8x^2$. Last, add the two rows together, and we get the answer $8x^2+22x-21$.</td>
</tr>
</table>

<table>
<tr>
<td rowspan="2">2
【多項式的除法】
Dividing polynomials</td>
<td>Example 8
Calculate $(5x^2-2x-6)\div(x+3)=$?
$5x^2\div x=5x$
$5x-17$ → $(-17x)\div x=-17$
$x+3\,)\,5x^2-\ 2x-\ 6$
$5x^2+15x$ → $(x+3)\times 5x=5x^2+15x$
$-17x-\ 6$
$-17x-51$ → $(x+3)\times(-17)=-17x-51$
45
Therefore, we get $(5x^2-2x-6)\div(x+3)=5x-17\cdots 45$</td>
</tr>
<tr>
<td>To calculate $(5x^2-2x-6)\div(x+3)$, we still use the vertical method, or we call it long division. In the beginning, write down the dividend and divisor, and we have to divide the first term of the dividend by the first term of the divisor. That is, divide $5x^2$ by x, and we get $5x$. Write above $5x$ the division line, and multiply the entire divisor by $5x$. $(x+3)$ times $5x$ equals $5x^2+15x$, and write it below the dividend. Then, subtract two rows, we get $-17x$. Bring down the -6 to obtain $-17x-6$, and we can repeat the same process. $-17x$ divided by x equals -17, and multiply $x+3$ by -17, which is $-17x-51$. Writing down the result below $-17x-6$ and subtracting two rows, we have 45. The degree of divisor is 1 and the degree of 45 is 0. Because the degree of 45 is less than the degree of the divisor, we complete the process. The quotient is $5x-17$, and the remainder is 45.</td>
</tr>
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