Matrix II

I. Key mathematical terms

Terms	Symbol	Chinese translation
Reciprocal		
Inverse Matrix		
Transpose		

II. Inverse Matrix

In real numbers, if $a \neq 0$ there exists a unique **reciprocal** $b = \frac{1}{a}$ which satisfies "$ab = ba = 1$".

Definition:(Inverse Matrix)

Let A be a $n \times n$ square matrix. We say A is invertible (or nonsingular) if there exists a $n \times n$ square matrix B such that

$$AB = BA = I_n$$

If such matrix B exists, we say B is "the inverse of A" and denoted as $B = A^{-1}$.
If no such inverse B exists for a matrix A, we say A is non-invertible (or singular).
<key> There can only exist at most one inverse.
<key>The matrix is singular if and only if det(A) = 0.

Example1.

$$A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$$

For each of the matrices A and B determine whether the matrix is singular. If the matrix is non-singular, find its inverse.

(Hint: You can use Cramer's rule to solve the linear equation.)

(1) $A = \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix}$

(2) $B = \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

Inverse of 2x2 Matrix

We can fine the inverse of any non-singular matrix.

The inverse of a matrix M is the matrix M^{-1} such that $MM^{-1} = M^{-1}M = I$.

In the case of 2x2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ a simple formula exists to find its inverse:

$$\text{If } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ then } A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

<key>

(1) If det(A)=0, then A is noninvertible.

(2) If det(A)≠0, then A is invertible.

<proof>

To varify the formula above, suppose $B = \begin{pmatrix} x & u \\ y & v \end{pmatrix}$ such that $AB = BA = I_2$.

$$AB = I_2$$

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & u \\ y & v \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$ We can get two systems of linear equations.

$$\Rightarrow \begin{cases} ax + by = 1 \\ cx + dy = 0 \end{cases} \text{ and } \begin{cases} au + bv = 0 \\ cu + dv = 1 \end{cases}$$ We can use Cramer's rule to solve these equations.

$\Rightarrow$ (1) $\begin{cases} ax + by = 1 \\ cx + dy = 0 \end{cases}$ (2) $\begin{cases} au + bv = 0 \\ cu + dv = 1 \end{cases}$

Finally we have:

For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A^{-1} = \frac{1}{\qquad}\begin{pmatrix} & \\ & \end{pmatrix} = \frac{1}{\qquad}\begin{pmatrix} & \\ & \end{pmatrix}$

Example2.

$$\text{Given } A=\begin{pmatrix}5 & 3\\ 2 & 1\end{pmatrix}, B=\begin{pmatrix}7 & -8\\ -4 & 5\end{pmatrix}$$

(1) Find the inverse matrix of A.

(2) If $AX=AB$, Find X

III. Solving systems of equations by using matrices

We can use the inverse of nxn matrix to solve a system of n simultaneous linear equations in n unknowns. We will introduce how to solve the 2 simultaneous linear equations in 2 unknowns in our class.

$$\text{If } A\begin{pmatrix}x\\ y\end{pmatrix}=v \text{ and } A \text{ is non-singular, then } \begin{pmatrix}x\\ y\end{pmatrix}=A^{-1}v.$$

Example3.

Use an inverse matrix to solve the simultaneous equations: $\begin{cases}2x-4y=4\\ 3x-5y=3\end{cases}$

Example4.

Matrix A is a 2x2 matrix which satisfies $A\begin{pmatrix}8\\ 5\end{pmatrix}=\begin{pmatrix}1\\ 2\end{pmatrix}, A\begin{pmatrix}3\\ 2\end{pmatrix}=\begin{pmatrix}-1\\ 1\end{pmatrix}$

(1) Find matrix A

(2) If $AB=\begin{pmatrix}3 & 0\\ 0 & 3\end{pmatrix}$, find B

IV. Properties of Inverse and Transpose

Definition:(Transpose of Matrix)

Given an $m \times n$ matrix W we define W^t, the transpose of W, to be the $n \times m$ matrix whose (i,j)th entry is the (j,i)th entry of W; that is the matrix for which

$$(W^t)_{ij} = W_{ji}$$

For example

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 2 & -1 & 3 \end{pmatrix} \quad A^t = \begin{pmatrix} 1 & 2 \\ 0 & -1 \\ -4 & 3 \end{pmatrix} \ ; B = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad B^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

<key> Transpose of matrix is found by interchanging rows into columns (or columns into rows).

Properties of Inverse and Transpose

(1) A martix has at most one inverse.

(2) Assume A and B are invertible $n \times n$ matrices. Then the product AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(3) We have $(AB)^t = B^t A^t$ and $(\lambda A)^t = \lambda A^t$ for any matrix A and real constant λ.

(4) Assume A is invertible. Then the transpose A^t is invertible and

$$(A^t)^{-1} = (A^{-1})^t$$

<explanation>

(1)

(2)

(3)

(4)

(Hint: Use the prorperties of inverse and transpose to solve example5~7.)

Example5.

Matrix A is a 2x2 matrix and $A^2=\begin{pmatrix}3 & 4\\ 8 & 11\end{pmatrix}, A^3=\begin{pmatrix}11 & 15\\ 30 & 41\end{pmatrix}$ Find A.

Example6.

Matrix A and B are invertible 2x2 matrices. Suppose $AB=\begin{bmatrix}3 & -4\\ -5 & 7\end{bmatrix}$. Find BA.

Example7.

Matrix A and B are invertible 2x2 matrices, such that $BAB=I$.

(1) Prove that $A=B^{-1}B^{-1}$. Given that $B=\begin{pmatrix}2 & 5\\ 1 & 3\end{pmatrix}$

(2) Find the matrix A such that $BAB=I$

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