一次與二次函數

Linear and Quadratic Functions

Material	Vocabulary
函數的定義 設 x 與 y 是兩個變數。當 x 的值給定時，y 的值也隨著 x 的值而唯一確定，我們稱這種對應關係為「y 是 x 的**函數**」。若將此函數命名為 f，則用記號 $y=f(x)$ 表示。	Linear Function (一次函數), Quadratic Function (二次函數), Variable (變數), Correspondence (對應關係).

Translations

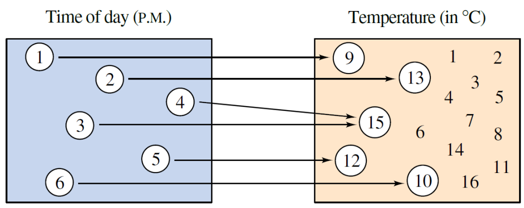
Let x and y be two variables. If the value of x is fixed and there exists exactly one value of y that depends on x, we call this kind of correspondence as “y is a function of x.” If we name the function f, we write “y is f of x”, or y=f(x) which is read “y equals f of x.”

Illustrations

A. Definition of Function

A function f from a set A to a set B is a relation that assigns to each element x in the set A to exactly one element y in the set B. The set A is the domain (or set of inputs) of the function f, and the set B contains the range (or set of outputs).

函數 f 是一個對應關係 $f:A\to B$ ，滿足所有在 A 集合裡的元素 x，存在唯一在集合 B 裡的元素 y，使得 f 將 x 對應到 y（ $^{\forall}x\in A,\ ^{\exists!}y\in B\ s.t.\ f(x)=y.$ ）。



Set A is the domain.
Inputs: 1, 2, 3, 4, 5, 6

Set B contains the range.
Outputs: 9, 10, 12, 13, 15

We can also use ordered pairs to represent this function as shown below. The first coordinate (x-value) is the input and the second coordinate (y-value) is the output.

$$\{(1,9),(2,13),(3,15),(4,15),(5,12),(6,10)\}$$

底下的數對可以代表這個函數，數對中的第一個值為 x，第二個值為 y。

B. Characteristics of a Function from Set A to Set B

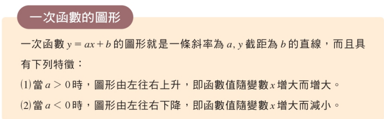
1. Each element in A must be matched with an element in B.
2. Some elements in B may not be matched with any element in A.
3. Two or more elements in A may be matched with the same element in B.
4. An element in A (the domain) cannot be matched with two different elements in B.

When a relationship does not follow those rules then it is not a function. It is still a relationship, just not a function.

函數 $A \to B$ 的特徵

1. A 集合裡的每個元素在 B 集合裡必須要有一對應的元素。
2. B 集合裡有些元素可能不會被對應到任何一個在 A 集合裡的元素。
3. 可能會有二個或以上個元素對應到同一個在 B 集合裡的元素。
4. A 集合裡的任一個元素不能對應到二個在 B 集合裡的元素。

若一對應關係不符合上述性質，則其關係不為函數，但仍然是一種對應關係，只是不被稱為函數。

Material	Vocabulary
 一次函數的圖形 一次函數 $y = ax + b$ 的圖形就是一條斜率為 a, y 截距為 b 的直線，而且具有下列特徵： (1) 當 $a > 0$ 時，圖形由左往右上升，即函數值隨變數 x 增大而增大。 (2) 當 $a < 0$ 時，圖形由左往右下降，即函數值隨變數 x 增大而減小。	Slope (斜率), Intercept (截距), Increase (增加), Decrease (減少), Slant (傾斜), Independent Variable (自變數), Dependent Variable (應變數/依變數).

Translations

Graphs of Linear Functions

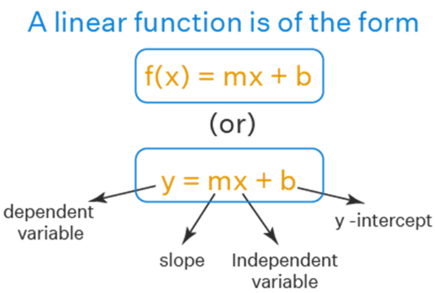
A **linear function** is of the form $y = ax + b$, where a is the slope of the line and b is the y-intercept of the line. The Graphs of Linear Functions have the following characteristics:

(1) When a is greater than zero, the graphs of the function slants up from left to right. The function is increasing, and the y-value increases as the x-value increases.

(2) When a is less than zero, the graphs of the function slants down from left to right. The function is decreasing, and the y-value decreases as the x-value increases.

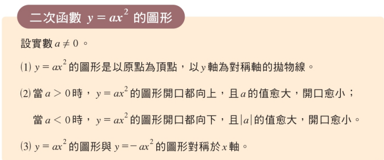
Illustrations

Definition of a Linear Function



$y = f(x) = mx + b$, with *m* and *b* being real numbers, *m* is not zero, is called a linear function involving one independent variable x and a dependent variable y. Sometimes this is shortened to read “a linear function of one variable,” meaning there is only one independent variable.

函數 $y = f(x) = mx + b$ ，其中 *m* 與 *b* 為實數，*m* 不為 0，稱此線性函數有一自變數 *x* 及一應變數 *y*。也可簡稱為一個變數的線性函數。

Material	Vocabulary
 二次函數 $y = ax^2$ 的圖形 設實數 $a \neq 0$。 (1) $y = ax^2$ 的圖形是以原點為頂點，以 y 軸為對稱軸的拋物線。 (2) 當 $a > 0$ 時， $y = ax^2$ 的圖形開口都向上，且 a 的值愈大，開口愈小； 當 $a < 0$ 時， $y = ax^2$ 的圖形開口都向下，且 $\|a\|$ 的值愈大，開口愈小。 (3) $y = ax^2$ 的圖形與 $y = -ax^2$ 的圖形對稱於 x 軸。	Vertex (頂點), Parabola (拋物線), Narrow (窄), Symmetric (對稱的), Upward (向上), Wide (寬), Downward (向下), Reflect (鏡射).

Translations

The graphs of quadratic function $y = ax^2$

(1) The vertex of the graphs that $y = ax^2$ (y equals a times x squared) is origin. The graph is the parabola which is symmetric about the y-axis

(2) If a is greater than zero, the graph of the quadratic function opens up and the larger of the value of a, the narrower the graph is.
If a is less than zero, the graphs of the quadratic function opens down and the larger of the absolute value of a, the narrower the graph is.

(3) The graph of y equals a time x squared and y equals negative a times x squared is symmetric to each other about the x-axis.

Notes:

The word “Quad” means “a square or rectangular outside area with buildings on all four sides.” So, “Quadratic” means “to make square”. In other words, a quadratic equation is an “equation of degree 2.”

“Quadratic”的字根為“Quad”其意思為“四”，而“Quadratic”意思為“做一方形”。也就是說“quadratic equation”為二次方程式。

Illustrations

Transform the Graph of $y = x^2$

A. The Parabola Opens Upward.

1. From $y=x^2$ to $y=2x^2$.

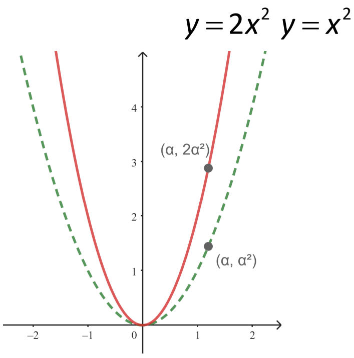


Figure 1

Multiplying a function by a constant stretches its graph vertically. Each point $(\alpha,2\alpha^2)$ on the graph of $y=2x^2$ is two times as far from the x-axis as the corresponding point (α,α^2) on $y=x^2$, as illustrated by *Figure 1*.

將函數乘一個常數會使其圖形垂直伸縮。如圖 1，每個在函數 $y=2x^2$ 圖形上的點 $(\alpha,2\alpha^2)$ 與 x 軸的距離，為對應在函數 $y=x^2$ 圖形上的點 (α,α^2) 與 x 軸的距離的 2 倍。

2. From $y=x^2$ to $y=ax^2$. ($a>0$)

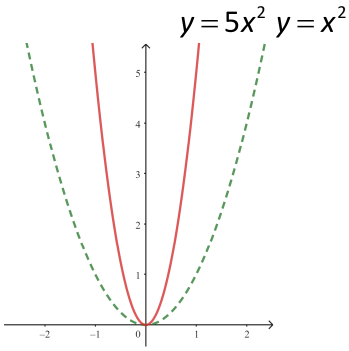


Figure 2

If $a>0$ in $y=ax^2$, the parabola opens upward. In this case, the y-coordinate of the vertex is the minimum value of the function, and the vertex is the lowest point of the parabola. A large positive number of the leading coefficient *a* makes a narrow parabola; a positive value of a positive value of *a* that is between 0 and 1 makes the parabola wide. As illustrated by *Figure 2*. the y-coordinate of the vertex is the minimum value of the function

若 $a>0$ ，函數 $y=ax^2$ 為一開口向上拋物線，頂點的 y 值為最小值（最低點）。領導係數 *a* 值越大導致開口越小；反之，*a* 值越接近 0 使其開口越大。圖 2 的 *a* 值為 5。

B. The Parabola Opens Downward.

1. From $y = x^2$ to $y = -x^2$.

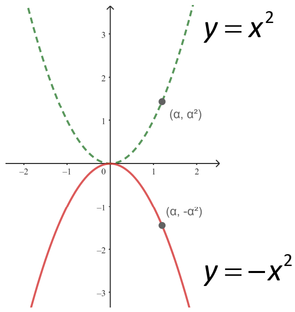


Figure 3

Each point on the graph of $y = -x^2$ has the same distance from the x-axis as the corresponding point on $y = x^2$. Furthermore, multiplying a function by a negative number reflects the graph across the x-axis, as illustrated by *Figure 3*. As a result, the graphs are symmetric about the x-axis.

如圖 3，每個在函數 $y = -x^2$ 圖形上的點 $\left(\alpha, -\alpha^2\right)$ 與 x 軸的距離，與對應在函數 $y = x^2$ 圖形上的點 $\left(\alpha, \alpha^2\right)$ 至 x 軸的距離相等。此外，將函數乘上負數會使圖形 x 軸鏡射。因此兩圖形對稱於 x 軸。

2. From $y = x^2$ to $y = ax^2$. ($a < 0$)

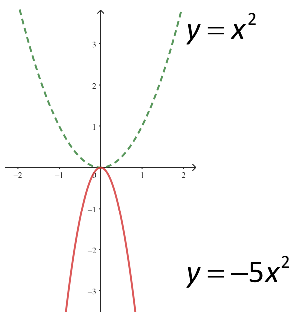
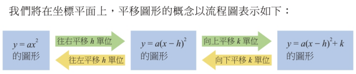


Figure 4

If $a < 0$ in $y = ax^2$, the parabola opens downward. In this case, the y-coordinate of the vertex is the maximum value of the function, and the vertex is the highest point of the parabola. Again, as the coefficient *a* becomes more negative, the parabola will become narrower. A value of *a* that is approaching 0 ($-1 < a < 0$) makes the parabola wider. As illustrated by *Figure 4*.

若 $a < 0$，函數 $y = ax^2$ 為一開口向下拋物線，頂點的 y 值為最大值（最高點）。領導係

數 a 值為越小的負數則開口越小；反之，a 值越接近 0 使其開口越大。圖 4 的 a 值為 –5。

Material	Vocabulary
	Shift (平移), Vertically (垂直地), Horizontally (水平地).

Sentences

1. Start with the equation $y = ax^2$. Replace every x by $x - h$ to give the new equation $y = a(x-h)^2$. This shifts the graph RIGHT by h units. (以 $y = ax^2$ 為起始方程式，將方程式中所有的 x 以 $x - h$ 取代得新方程式 $y = a(x-h)^2$。其圖形會向右平移 h 單位。)
2. Adding k to the previous y-values gives the new equation $y = a(x-h)^2 + k$. This shifts the graph UP by k units. (將前一方程式 $y = a(x-h)^2$ 加上 k 得新方程式 $y = a(x-h)^2 + k$，其圖形會向上平移 k 單位。)

illustrations

A. Shifts to the RIGHT/LEFT by *h* units.

When a constant h is added to, or subtracted from, the input x of a function, the corresponding graph is shifted $-h$ units to the right or left. (將自變數 x 加或減 h，其圖形會水平平移 $-h$ 個單位。)

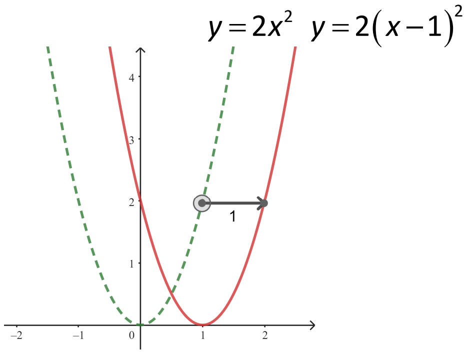


Figure 5

For example, move the graph of $y = 2x^2$ one unit to the right to create the graph of $y = 2(x-1)^2$, as illustrated by *Figure 5*. (如圖 5 所示，將圖形 $y = 2x^2$ 向右移 1 個單位，則方程式會變成 $y = 2(x-1)^2$。)

B. Shifts UP/DOWN by *k* units.

Adding or subtracting a constant k from a function affects its graph—all points on the graph are shifted k units up or down on the coordinate plane. (將方程式加或減 k，其圖形會垂直平

移 k 個單位。)

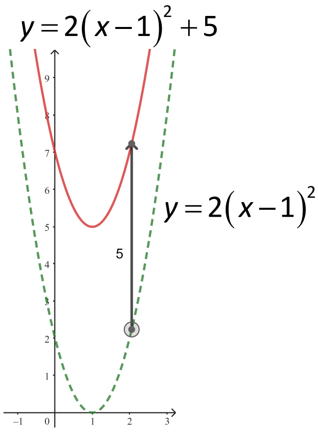


Figure 6

Move the graph of $y=2(x-1)^2$ five unit up to create the graph of $y=2(x-1)^2+5$, as illustrated by *Figure 6*. (如圖 6 所示，將圖形 $y=2(x-1)^2$ 向上移 5 個單位，則方程式會變成 $y=2(x-1)^2+5$ 。)

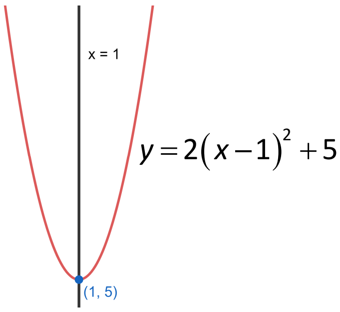
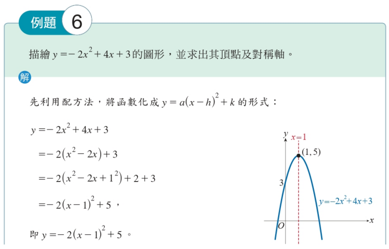


Figure 7

Thus, as *Figure 7* shown the vertex of the parabola $y=2(x-1)^2+5$ is $(1,5)$. The axis of symmetry equation of it is $x=1$. (如圖 7 所示，此拋物線的頂點 $(1,5)$ 為滿對稱軸為 $x=1$ 。)

Material	Vocabulary
 例題 6 描繪 $y=-2x^2+4x+3$ 的圖形，並求出其頂點及對稱軸。 解 先利用配方法，將函數化成 $y=a(x-h)^2+k$ 的形式： $y=-2x^2+4x+3$ $=-2(x^2-2x)+3$ $=-2(x^2-2x+1^2)+2+3$ $=-2(x-1)^2+5$， 即 $y=-2(x-1)^2+5$。 $x=1$　(1, 5)　3　$y=-2x^2+4x+3$　O　x　y	Sketch (勾畫), Standard Form (標準式), Vertex Form (頂點式), Completing the Square (配方), Discriminant (判別式).

Translations

Sketch: $y=-2x^2+4x+3$. Find the axis of symmetry and the vertex of the graph.

Factor out -2 only from the terms with the variable *x*. (從 x 項提出係數 -2 。)

$$y=-2(x^2-2x)+3$$

The coefficient of the linear term inside the parenthesis is -2. Divide it by 2 and square it, which is 1^2. Add the value 1^2 inside the parenthesis. To make the original equation the same, we add $-1^2\times(-2)=2$ outside the parenthesis. (括號裡 x 一次項係數為-2，將其除 2 且平方，得到 1。在括號裡加 1，並為了讓式子相等，再括號外須加 2。)

$$y=-2\left(x^2-2x+1^2\right)+3+2$$

Finally, express the trinomial inside the parenthesis as the square of a binomial and then simplify the outside constants. (最後，將括號裡的三項式表示成完全平方式，並化簡常數項。)

$$y=-2(x-1)^2+5$$

It is now in the vertex form $y=a(x-h)^2+k$ where the vertex (h,k) is $(1,5)$. The axis of symmetry is that x equals x-coordinate of vertex which is $x=1$. (現在我們已將式子表示為頂點式，因此頂點為$(1,5)$，對稱軸為 x 等於頂點 x 坐標，$x=1$。)

illustrations

The Vertex formula of a parabola is used to find the coordinates of the point where the parabola crosses its axis of symmetry. As we know, the standard equation of a parabola is $y=ax^2+bx+c$. (頂點式可以拿來找頂點及對稱軸，而一般式則形如 $y=ax^2+bx+c$。)

If the coefficient a is positive then the vertex is the bottom of the U- shaped curve and if it is negative the vertex point is the top of the U-shaped curve. (如果 a 是正數，則其頂點在 U 形曲線的底下；反之，若 a 為負數，頂點則在 U 形曲線的上方。)

Let's convert the standard form of a parabola $y=ax^2+bx+c$ to the vertex form $y=a(x-h)^2+k$ by completing the squares. (接著利用配方法使一般式轉換成頂點式，以找出頂點的公式。)

$$\begin{aligned} y&=ax^2+bx+c \\ &=a\left(x^2+\frac{b}{a}x\right)+c \\ &=a\left(x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2\right)+c-\frac{b^2}{4a} \\ &=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a} \end{aligned}$$

Thus, the vertex formula is $(h,k)=\left(\frac{-b}{2a},\frac{-D}{4a}\right)$, where $D=b^2-4ac$ is the discriminant. (因此，頂點的公式為$(h,k)=\left(\frac{-b}{2a},\frac{-D}{4a}\right)$，其中$D=b^2-4ac$為判別式。)

Summary

The graph of quadratic function $y=ax^2+bx+c$ is a parabola, and its characteristic are as follows.

1. When a is positive, the parabola opens up and the lowest point is the vertex.
   When a is negative, the parabola opens down and the highest point is the vertex.
2. The axis of the symmetry equation of a parabola is $x=\frac{-b}{2a}$ with vertex $\left(\frac{-b}{2a},\frac{-D}{4a}\right)$, where $D=b^2-4ac$.

二次函數$y=ax^2+bx+c$的圖形是一條拋物線，而且具有下列特徵：

1. 當$a>0$，拋物線的開口向上，頂點是圖形的最低點。
   當$a<0$，拋物線的開口向下，頂點是圖形的最高點。
2. 此拋物線的對稱軸為直線$x=\frac{-b}{2a}$，頂點坐標為$\left(\frac{-b}{2a},\frac{-D}{4a}\right)$，其中$D=b^2-4ac$。

Material

(五)二次函數圖形的分類

上一單元中，我們知道：二次函數$y=ax^2+bx+c$的圖形的開口向上或向下，可由係數a的正負判定。若要進一步知道圖形與x軸的相交情形，則可以藉助判別式$D=b^2-4ac$的值來判定。說明如下：

首先考慮$a>0$的情形，此時二次函數$y=ax^2+bx+c$的圖形是開口向上，且頂點為

$$\left(-\frac{b}{2a},-\frac{b^2-4ac}{4a}\right)=\left(-\frac{b}{2a},-\frac{D}{4a}\right)$$

的拋物線。

(1) 當$D>0$時，頂點的y坐標$-\frac{D}{4a}<0$，此時頂點在x軸的下方，圖形與x軸交於兩點。

(2) 當$D=0$時，頂點的y坐標$-\frac{D}{4a}=0$，此時頂點落在x軸上，圖形與x軸相切。

(3) 當$D<0$時，頂點的y坐標$-\frac{D}{4a}>0$，此時頂點在x軸的上方，圖形與x軸不相交。

Translations

The Discriminant from a Graph

If we want to identify how many x-intercepts of the quadratic function are, we could use discriminant to find the numbers.

Considering $a>0$, the parabola $y=ax^2+bx+c$ opens up and the vertex is $\left(\frac{-b}{2a},\frac{-D}{4a}\right)$.

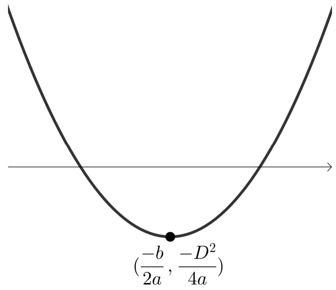


Figure 8

1. When the discriminant is positive, the y-coordinate of the vertex $\frac{-D}{4a}$ is negative. That means the vertex is under the x-axis, so the graph has two *x*-intercepts. Illustrated by *Figure 8*.

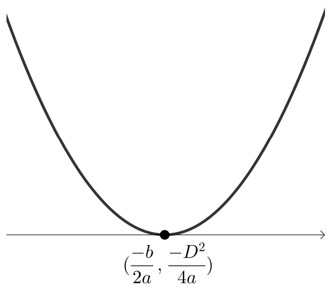


Figure 9

2. When the discriminant is zero, the y-coordinate of the vertex $\frac{-D}{4a}$ is also zero. That means the vertex is on the x-axis, so the graph has exactly one *x*-intercept. Illustrated by *Figure 9*.

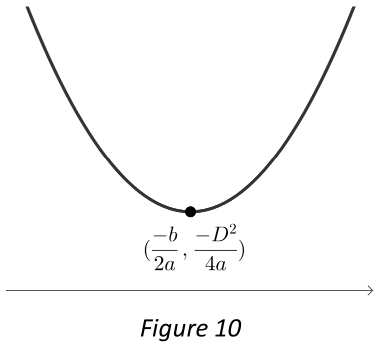
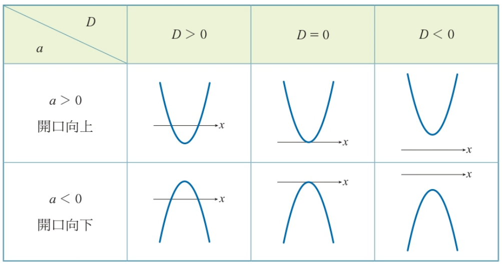


Figure 10

3. When the discriminant is negative, the y-coordinate of the vertex $\frac{-D}{4a}$ is positive. That means the vertex is above the x-axis, so the graph has no *x*-intercept. Illustrate by *Figure 10*.

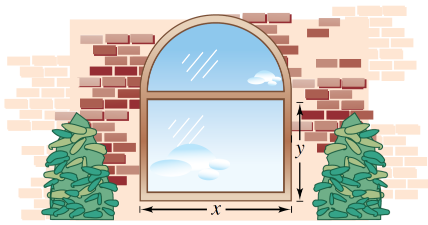
As the scenario of $a<0$, we could use the same way to discuss and get the result as follows.

至於 $a<0$ 的情形，可仿照以上的方法討論。得到二次函數圖形的分類如下：



Supplementary Materials

Maximum Area A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window (see figure). The perimeter of the window is 16 feet.



(a) Write the area *A* of the window as a function of *x*.

(b) What dimensions produce a window of maximum area?

Solution

The perimeter is 16 feet, and from the figure we have $P = y + x + y + \frac{1}{2} \times 2\pi \times \frac{x}{2}$.

It gives $2y + \left(1 + \frac{\pi}{2}\right)x = 16$. Isolate *y*. $y = 8 - \left(\frac{2+\pi}{4}\right)x$ ...(1)

(a) We find the area, and substitute in for *y* with (1).

$$\begin{aligned} A &= xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 \\ &= x\left[8 - \left(\frac{2+\pi}{4}\right)x\right] + \frac{\pi}{8}x^2 \\ &= -\left(\frac{\pi+4}{8}\right)x^2 + 8x \end{aligned}$$

(b) Which shows that $a = -\left(\frac{\pi+4}{8}\right)$ and $b = 8$. Because $a<0$, the function has a maximum at $x = \frac{-b}{2a}$. So, the area reaches its maximum when the dimensions of the window with the width is

$$x=\frac{-b}{2a}=\frac{-8}{2\left[-\left(\frac{\pi+4}{8}\right)\right]}=\frac{32}{\pi+4}\approx 4.48\ \text{ft.}$$

and the length is

$$y=8-\left(\frac{2+\pi}{4}\right)\left(\frac{32}{\pi+4}\right)\approx 2.24\ \text{ft.}$$

The maximum area is $A=-\left(\frac{\pi+4}{8}\right)x^2+8x=-\left(\frac{\pi+4}{8}\right)\left(\frac{32}{\pi+4}\right)^2+8\times\frac{32}{\pi+4}=\frac{128}{\pi+4}\approx 17.92\ \text{ft}^2$.

Translations

Vocabulary: Construct (建造), Adjoin (鄰接), Semicircle (半圓), Perimeter (周長), Rectangular (矩形的), Dimension (尺寸), Feet(複數英呎，單數為 foot，簡寫為 ft.), Square Feet (平方英呎，簡寫為 sq. ft. /ft^2.).

Sentences:

1. Substitute in for *y*. (代入 y 值。)
2. The area reaches its maximum when... (在...的情況下，面積達到最大值。)
3. The function has a maximum at $x=\frac{-b}{2a}$. (函數有最大值，當 $x=\frac{-b}{2a}$ 。)

References

1. 許志農、黃森山、陳清風、廖森游、董涵冬（2019）。*數學 1：單元 9 一次與二次函數*。龍騰文化。
2. Barbara Lee Bleau (2003). *Forgotten Algebra Third Edition.* Barron's.
3. Ron Larson & Robert P. Hostetler (2001). *Algebra and Trigonometry Fifth Edition.* Houghton Mifflin Company.
4. Ron Larson (2018). *Precalculus with CalcChat and CalcView Tenth Edition.* Cengage Learning.
5. Cuemath (2023, Feb 1). *Quadratic Equation.* https://reurl.cc/aa2AGD.
6. Varsitytutor (2023, Feb 1). *Parabolas.* https://reurl.cc/10DjMX.
7. Chilimath (2023, Feb 1). *Complete the Square.* https://reurl.cc/VRKlEN.

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