Equation of a Line in 3D Space

I. Key mathematical terms

Terms	Symbol	Chinese translation
Direction vector (direction numbers) of a line		
Parametric equation of a line		
Symmetric equation of a line		

II. Equation of a Line in 3D Space

A line can be viewed as the set of all points in space that satisfy two criteria:

(i) They contain a particular point P, which we identify by a position vector x_0.

(ii) The vector between P and any other point on line Q is parallel to a given vector v.

As we've learned before, a line in the xy-plane is determined by a point on the line and the direction of the line. (Its slope/gradient or angle of inclination.) The equation of the line can be written using the point-slope form, slope-intercept form, and intercept form.... How about the line in three-dimensional(3D) space?

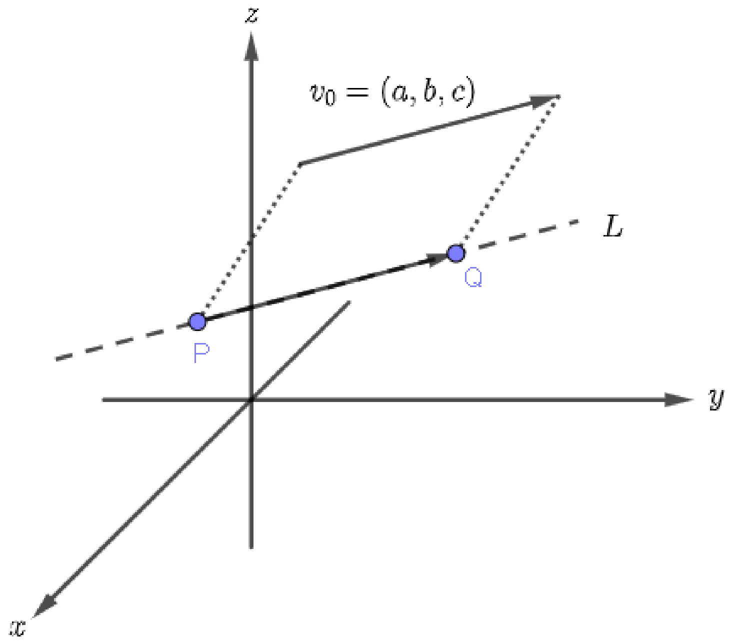
Likewise, a line L in three-dimensional space can be determined by a point $P(x_0, y_0, z_0)$ on L and the direction of L. In three dimensions, the direction of a line is conveniently described by a vector, so we let $v_0 = (a, b, c)$ be a vector parallel to L. Let $Q(x, y, z)$ be any arbitrary point on L, we find:

$$\overrightarrow{PQ} = (x - x_0, y - y_0, z - z_0) = (ta, tb, tc) = tv_0, t \in \mathbb{R}$$

Hence we have:

$$\begin{cases} x - x_0 = at \\ y - y_0 = bt \\ z - z_0 = ct \end{cases} \Rightarrow \quad \begin{matrix} x = x_0 + at \\ y = y_0 + bt, t \in \mathbb{R} \\ z = z_0 + ct \end{matrix}$$

We can represent these points and vectors in the following figure:



Parametric equation of a line in 3D space

The **parametric equation of a line in space** can be represented by a nonunique set of three equations of the form:

$$L: \begin{cases} x = x_0 + at \\ y = y_0 + bt, t \in \mathbb{R} \\ z = z_0 + ct \end{cases}$$

Where (x_0, y_0, z_0) is the coordinate of a point that lies on the line, (a, b, c) is a direction vector of the line, and t is a parameter that can be any real number.

Example1

Find the parametric equation of the line that passes through the given point and direction vector:

(1) Point $(-1, 2, 3)$, direction vector $(2, 3, 5)$

(2) Point $(-1, 0, 2)$, direction vector $(0, -1, 3)$

Example2

Find the parametric equation of the line that passes through points $(2,5,7)$ and $(-2,0,3)$. (Hint: You should find the direction vector by the given points first.)

Symmetric equation of a line in 3D space

We can represent a line by a parametric equation:

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct \quad t \in \mathbb{R}$$

If we solve each of the equations for t assuming a, b, and c are nonzero, we can have a different description of the same line:

$$\frac{x - x_0}{a} = t, \quad \frac{y - y_0}{b} = t, \quad \frac{z - z_0}{c} = t$$

This is the symmetric equation of a line in 3D space.

The **symmetric equation of a line in space** can be represented by the following:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Where (x_0, y_0, z_0) is the coordinate of a point that lies on the line, (a,b,c) is a direction vector of the line. ($abc \neq 0$) This form of the equation is closely related to the set of parametric equations.

Example3

Find the symmetric equation of a line that passes through points $(-2,4,7)$ and $(1,2,5)$. (Hint: You should find the direction vector by the given points first.)

Example4 (Converting a symmetric equation to a parametric equation.)

Find the parametric equation of the straight line $\frac{3x-7}{2}=\frac{y+5}{-1}=\frac{1-2z}{3}$.

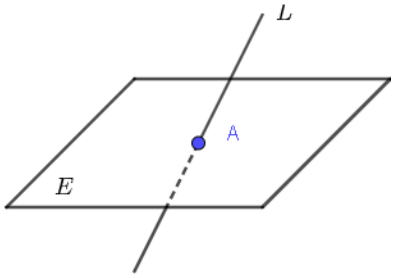
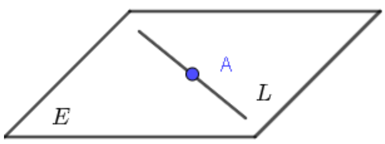
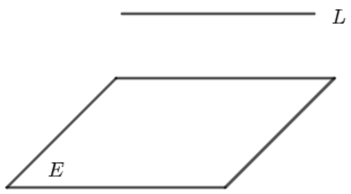
Example5 (Intersection of planes)

Find the parametric equation of the line of intersection between the two planes $E_1: x+3y-z+4=0$ and $E_2: 2x+5y+z+1=0$.

The relationships between a line and a plane

There are three possibilities that may occur when a line and a plane interact with each other, see the details in the following table:

The relationships between a line and a plane

Feature	Image	Description
Line intersect plane		The line intersects the plane at **<u>one point</u>**.
Plane containing line (Line lies on the plane)		The line intersects the plane at **<u>infinitely many points</u>**.
Parallel line and plane		The line intersects the plane at **<u>no point</u>**.

If you want to define the relationships between a line and a plane. We've broken down the steps needed below:

(1) Write the equation of the line in the parametric form.

$$(x, y, z) = (x_0 + at,\ \ y_0 + bt,\ \ z_0 + ct)\ \ t \in \mathbb{R}$$

(2) Write the equation of the plane in its scalar form.

$$E : ax + by + cz + d = 0$$

(3) Use *x, y, z*'s corresponding parametric equations to rewrite the scalar equation of the plane. Solve the equation for *t*.

(4) If *t* has exactly one solution, then the relationships will be line intersect plane. If *t* vanish (*t* can be any real number), then the relationships will be plane-containing line. If *t* has no solution, then the relationships will be parallel line and plane.

Example6

Determine the relationships between plane $E : 2x - 3y - 5z + 9 = 0$ and line L_1, L_2, L_3.

(1) $L_1 : \begin{cases} x = 6 + t \\ y = 2 - t\ , t \in \mathbb{R} \\ z = 1 + 2t \end{cases}$ (2) $L_2 : \begin{cases} x = 1 + s \\ y = 2 - s, s \in \mathbb{R} \\ z = 1 + s \end{cases}$ (3) $L_3 : \begin{cases} x = 3 + m \\ y = 1 - m\ , m \in \mathbb{R} \\ z = 2 + m \end{cases}$

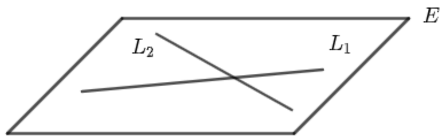
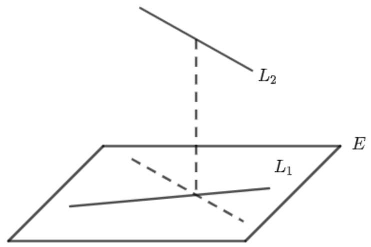
Example7

Find the projection of point $A(-4, 0, -6)$ onto the plane $E : 3x - y + 2z - 4 = 0$.

The relationships between two lines

Now we’ll talk about the relationships between two lines. Like the relationships between a line and a plane, the relationships between two lines also has three possibilities. Let’s see the details in the following table:

The relationships between two lines

Name	Image	Description
Parallel lines		Two lines lie in the same plane and has **<u>no intersections</u>**.
Intersecting lines		Two lines lie in the same plane and **<u>intersect at one point</u>**.
Skew lines		Two lines do not lie in the same plane and **<u>has no intersections</u>**.

If you want to define the relationships between two lines. We’ve broken down the steps needed below:

(1) Write the equation of lines in the parametric form.

$$L_1 : P(x,y,z) = (x_0 + a_0 t,\ \ y_0 + b_0 t,\ \ z_0 + c_0 t)\ \ t \in \mathbb{R}$$
$$L_2 : Q(x,y,z) = (x_1 + a_1 s,\ \ y_1 + b_1 s,\ \ z_1 + c_1 s)\ \ s \in \mathbb{R}$$

(2) Suppose $P = Q,$ solve the value of (t,s).

(3) If (t,s) has exactly one solution, then the relationships will be intersecting lines. If not, use the direction vector to check the relationships, there are three different cases:

Case1: L_1, L_2 have same direction vector and don’t intersect each other, then it will be two parallel lines.

Case2: L_1, L_2 have same direction vector and intersect each other, then it will be two coincident lines.

Case3: L_1, L_2 have different direction vector, then it will be two skew lines.

Example8

Determine the relationships between the two lines:

(1) $L_1: \frac{x+2}{1} = \frac{y-3}{2} = \frac{z+3}{-2}, L_2: \frac{x-5}{-3} = \frac{y+3}{4} = \frac{z+7}{1}$

(2) $L_1: \frac{x+2}{1} = \frac{y-3}{2} = \frac{z+3}{-2}, L_2: \frac{x-2}{-3} = \frac{y+2}{4} = \frac{z}{1}$

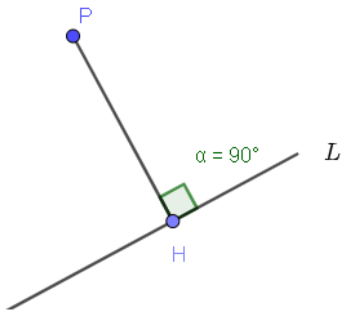
Example9

Determine the relationships between the two lines, if they intersecting each other, find the intersection of these two lines:

$$L_1: \frac{x-1}{2} = \frac{y+5}{4} = \frac{z+1}{1}, L_2: \begin{cases} x = 1+4t \\ y = 1+2t \ , t \in \mathbb{R} \\ z = -2+4t \end{cases}$$

The distance questions about lines

I. Distance between a point and a line



We've talked about the formula of distance between a point and a line on plane, but in space we should calculate in a different wat. Let's see the following example:

Example10

Find the distance between point $P(-5,0,-8)$ and line $L:\dfrac{x-3}{1}=\dfrac{y-2}{-2}=\dfrac{z+1}{2}$

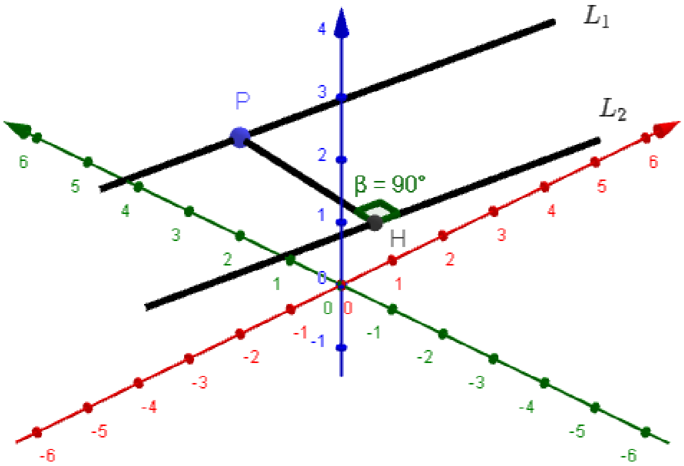
<sol>

1. Write the equation of line in the parametric form: $Q(3+t,2-2t,-1+2t), t\in\mathbb{R}$.
2. Find the vector $PQ=(8+t,2-2t,7+2t), t\in\mathbb{R}$.
3. Vector PQ is perpendicular to the directional vector of L. We can use the inner product to find t: $(8+t,2-2t,7+2t)\bullet(1,-2,2)=0$, $t=-2$.
4. Plug $t=-2$ into $\overline{PQ}=\sqrt{(-5-1)^2+(0-6)^2+(-8+5)^2}=\sqrt{81}=9$.

Example11

Find the distance between point $P(1,2,3)$ and line $L:\dfrac{x-6}{1}=\dfrac{y}{-4}=\dfrac{z-6}{2}$

II. Distance between two parallel lines



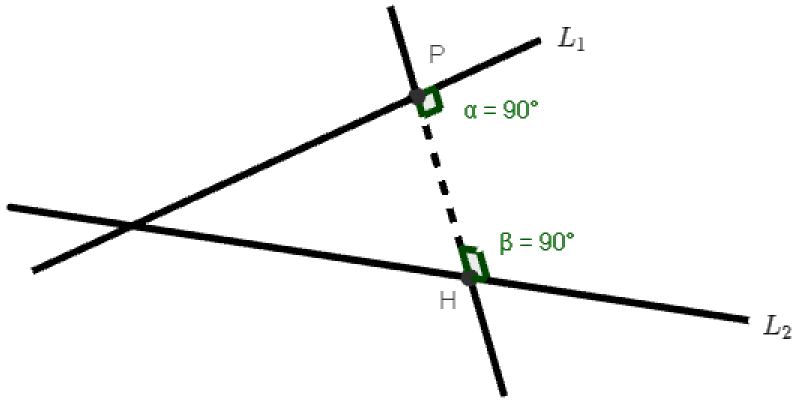
To find the distance between two parallel lines L_1, L_2, you only need to pick a point P on L_1 and find the distance between P and L_2 then you can get the distance between these two parallel lines.

Example12

Find the distance between two parallel lines:

$$L_1: \frac{x+1}{2} = \frac{y-1}{2} = \frac{z}{1}, L_2: \frac{x-1}{2} = \frac{y}{2} = \frac{z+2}{1}$$

III. Distance between two skew lines



To find the distance between two skew lines, we broke down the steps needed below:

(1) Write the equation of lines in the parametric form.

$$L_1 : P(x, y, z) = (x_0 + a_0 t,\ \ y_0 + b_0 t,\ \ z_0 + c_0 t)\ \ t \in \mathbb{R}$$

$$L_2 : Q(x, y, z) = (x_1 + a_1 s,\ \ y_1 + b_1 s,\ \ z_1 + c_1 s)\ \ s \in \mathbb{R}$$

(2) Suppose vector PQ normal to the direction vector of L_1 and L_2.

(3) Use the inner product $\begin{cases} \overrightarrow{PQ} \cdot (a_0, b_0, c_0) = 0 \\ \overrightarrow{PQ} \cdot (a_1, b_1, c_1) = 0 \end{cases}$ to solve (t, s).

(4) Plug the result of (t, s) into $\overline{PQ}$ to find the distance between these two skew lines.

Now, let's try the following example:

Example13

Two skew lines $L_1 : \frac{x+2}{1} = \frac{y-3}{2} = \frac{z+3}{-2}, L_2 : \frac{x-2}{-3} = \frac{y+2}{4} = \frac{z}{1}$

(1) Find the distance between these two lines

(2) Find the line which is perpendicular to both L_1 and L_2

<資料來源>

1. **Equation of line in 3D space**
   https://math.libretexts.org/Bookshelves/Calculus/Calculus_(OpenStax)/12%3A_Vectors_in_Space/12.05%3A_Equations_of_Lines_and_Planes_in_Space
   https://byjus.com/maths/equation-line/
   https://www.nagwa.com/en/explainers/365140723017/
   https://openstax.org/books/calculus-volume-3/pages/2-5-equations-of-lines-and-planes-in-space

2. **Pearson Edexcel AS and A level Mathematics Pure Mathematics Year 2**

3. 南一書局數學 **4A**

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