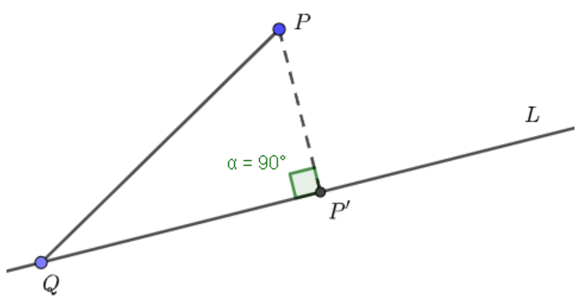
Applications of Linear Equations

I. Key mathematical terms

Terms	Symbol	Chinese translation
Projection		
Reflection		
Half-Plane		
Two-Variable Inequalities		

II. Distance between a point and a line

How do we find the distance between a point and a line? Considering the given point P (P does not lie on L.) and the line L, we can draw the graph:



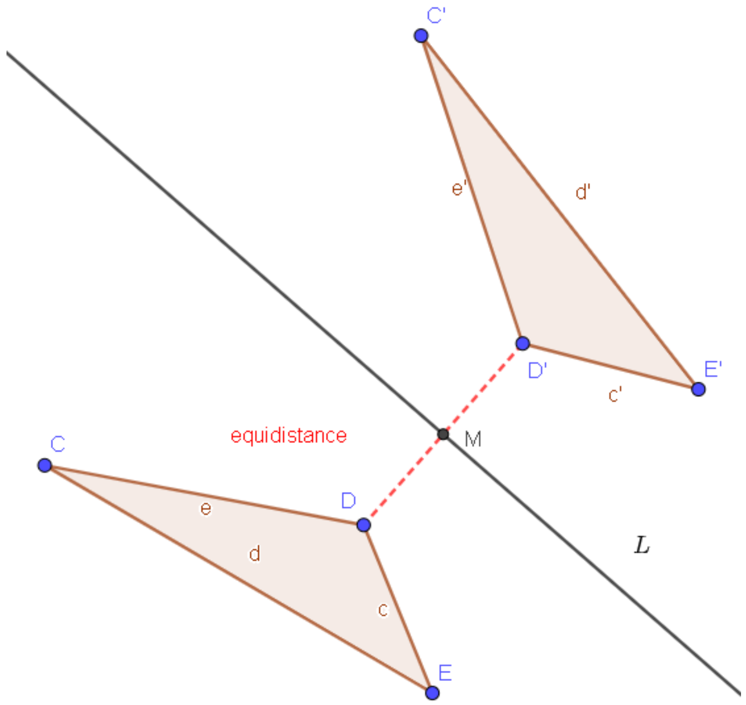
$\overline{PP'}$ is the distance from P to L and $\overline{PP'}$ is perpendicular to L.

$\overline{PQ}$ is not the distance from point P to line L.

To find out the distance between a point and a line(of the distance from a point to a line), we need to know the concept of reflection of points and projection of points.

Reflection of points

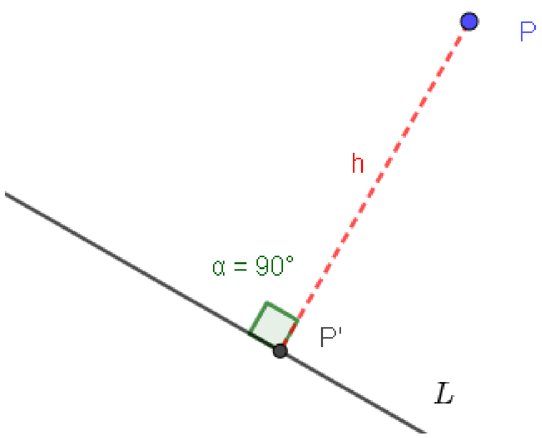
An image reflects through a line, is known as the line reflection. To find the distance from a point to a line we'll use line reflection. In Geometry, a reflection is known as a flip. A reflection is a mirror image of the shape (or a point). If we reflect a point P through line L to get another point P', these two points have equidistance to line L.



Reflection through line L.

Projection of points

Let L be a ~~straight~~ line. The (orthogonal) projection of a point P onto the line L is the point P'. The line segment $\overline{PP'}$ is orthogonal (vertical/perpendicular) to line L. We say that P has been projected onto line L.



P' is the projection of P onto line L.

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Example1.

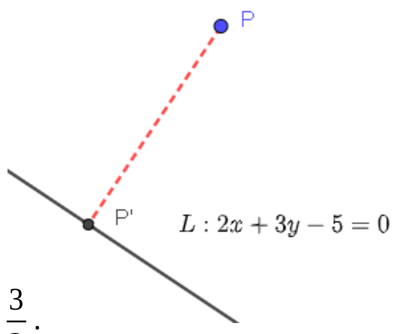
1. Consider point $A(5,-2)$
   (1) Find the reflection point of A over the x-axis.
   (2) Find the reflection point of A over the y-axis.
   (3) Find the reflection point of A over the line $x=y$.

2. Consider point $B(-1,3)$
   (1) Find the projection point of B onto the x-axis.
   (2) Find the projection point of B onto the y-axis.

3. Consider point $C(7,2)$
   (1) Find the reflection point of C over the line $x=y$.
   (2) Find the projection point of C onto the line $x=y$.

Distance between a point and a line

Example2.



Consider a point $P(3,4)$ and a line $L: 2x+3y-5=0$. Find:

(1) The projection point of P onto the line $L: 2x+3y-5=0$.

(2) The distance of P from line L.

<sol>

(1) By graph, the slope of L is $-\dfrac{2}{3}$ so the slope of $\overleftrightarrow{PP'}$ is $\dfrac{3}{2}$.

We can use the point-slope form to have the line equation

$\overleftrightarrow{PP'}: y-4=\dfrac{3}{2}(x-3) \Rightarrow \overleftrightarrow{PP'}: 3x-2y-1=0$ solve the system of equations

$$\begin{cases} 2x+3y-5=0 \\ 3x-2y-1=0 \end{cases} \Rightarrow Q(1,1).$$

(2) The distance of P from line L is $\overline{PP'}=\sqrt{(3-1)^2+(4-1)^2}=\sqrt{13}.$

If we want to find the distance between a point and a line in a more efficient way, we'll need a formula. There are many ways we can derive the formula to measure the distance of the point from a line. (We can use the equation of lines, vector, distance formula, and the area of the triangle formula…) Here we'll use the distance formula and the equation of lines to derive the distance formula of a point from a line.

Distance between a point and a line

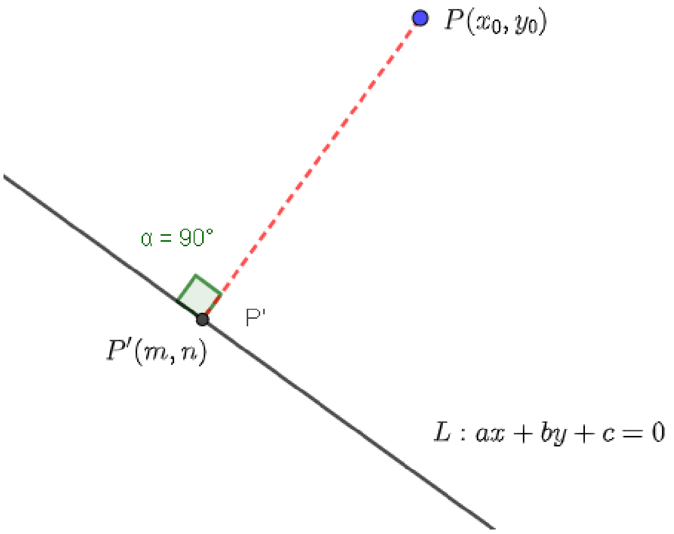
The distance between the point (x_0, y_0) and the line $L: ax+by+c=0$ is

$$d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

<proof>

For simplicity, assume that the given line is neither horizontal nor vertical.

Consider the graph below:



Rewrite the equation $L: ax+by+c=0$ into the slope-intercept form.

$$y = -\frac{a}{b}x - \frac{c}{b}$$

It shows that the slope of L is $-\frac{a}{b}$ and $\overleftrightarrow{PP'}$ is vertical to L, so the slope of

$\overleftrightarrow{PP'}$ is $\frac{b}{a} = \frac{n-y_0}{m-x_0}$. Hence we have $\begin{cases} m - x_0 = at \\ n - y_0 = bt \end{cases}$, $P'(m,n) = (x_0+at, y_0+bt)$.

Now we apply the distance formula of two points:

$$\overline{PP'} = \sqrt{[(x_0+at)-x_0]^2 + [(y_0+bt)-y_0]^2} = \sqrt{(a^2+b^2)t^2} = |t|\sqrt{a^2+b^2} \ -(*)$$

For P' lies on L, we have:

$$a(x_0+at)+b(y_0+bt)+c=0 \Rightarrow t=-\frac{ax_0+by_0+c}{a^2+b^2} - (**)$$

By $(*)$ and $(**)$ we have:

$$\overline{PP'}=\sqrt{a^2+b^2}\times\left|-\frac{ax_0+by_0+c}{a^2+b^2}\right|=\frac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}.$$

Example3.

Use the formula of distance between a point and a line to find the distance between $A(-4,3)$ and the following straight lines.

(1) x-axis

(2) y-axis

(3) $3x-2y+5=0$

Example4.

Find the distance between $L_1: 4x-3y-18=0, L_2: 4x-3y+7=0$.

(Hint: You can pick any point P on L_1 and find the distance between P and L_1.)

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Distance between two lines

To calculate the distance between two lines, we can use the same formula of the distance between a point and a line. We can pick the point on one of the lines and find the distance between this point and another line.

> **Distance between two lines**
>
> The distance between $L_1 : ax + by + c_1 = 0, L_2 : ax + by + c_2 = 0$ is
>
> $$d(L_1, L_2) = \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}$$

(To prove this formula, you can use the same method mentioned in example4.)

Example5.

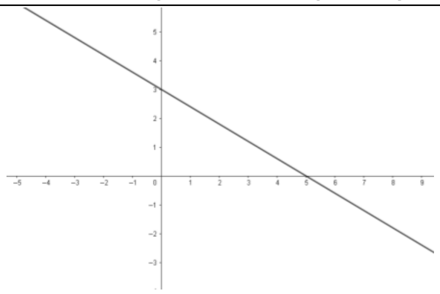
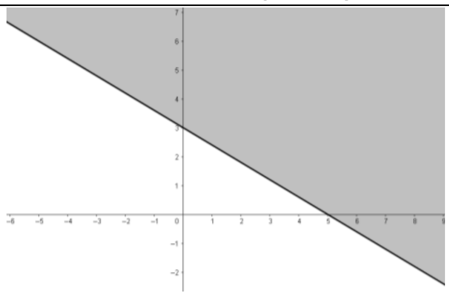
Use formula to find the distance between $L_1 : x - 3y - 5 = 0, L_2 : 2x - 6y + 10 = 0$.

Example6.

Find the area of the triangle with vertices $A(-1,1), B(2,-3), C(3,4)$.

III. Half-plane and two-variable inequalities

We know that a linear equation with two variables has infinitely many solutions that form a line. A linear inequality with two variables has a solution set consisting of a region that defines half of the plane.

Linear Equation (Equality)	**Linear Inequality**
 $3x + 5y = 15$	 $3x + 5y \geq 15$

For the inequality, the line defines the boundary of the region that is shaded. This gives that the ordered pair in the shaded region including the boundary line will satisfy the inequality.

Example7.

Check whether the following points is the solution of the inequality $4x-7y+6>0$.

(1) *A(1,3)*

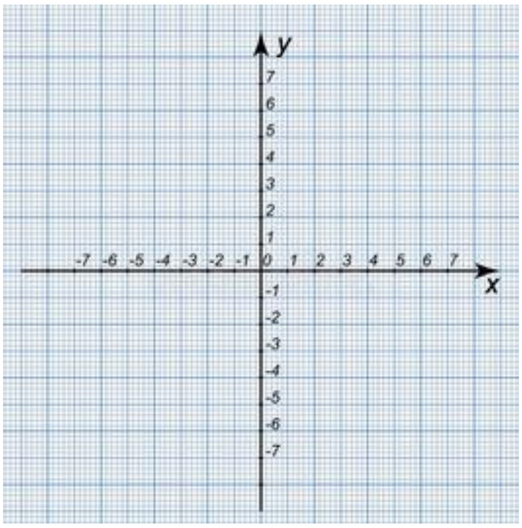
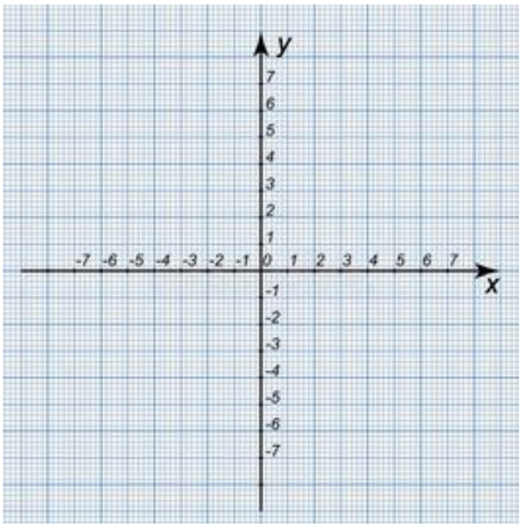
(2) *B(-2,-2)*

(3) *C(5,1)*

Solutions to linear inequalities are a shaded half-plane, bounded by a solid line(實線) or a dashed line(虛線). If we are given a strict inequality, we use a dashed line to indicate that the boundary is not included. If we are given an inclusive inequality, we use a solid line to indicate that the boundary is included.

Example8.

Graph the solution set of the following inequalities:

$2x-3y+6\le 0$	$3y-18>-6$
	

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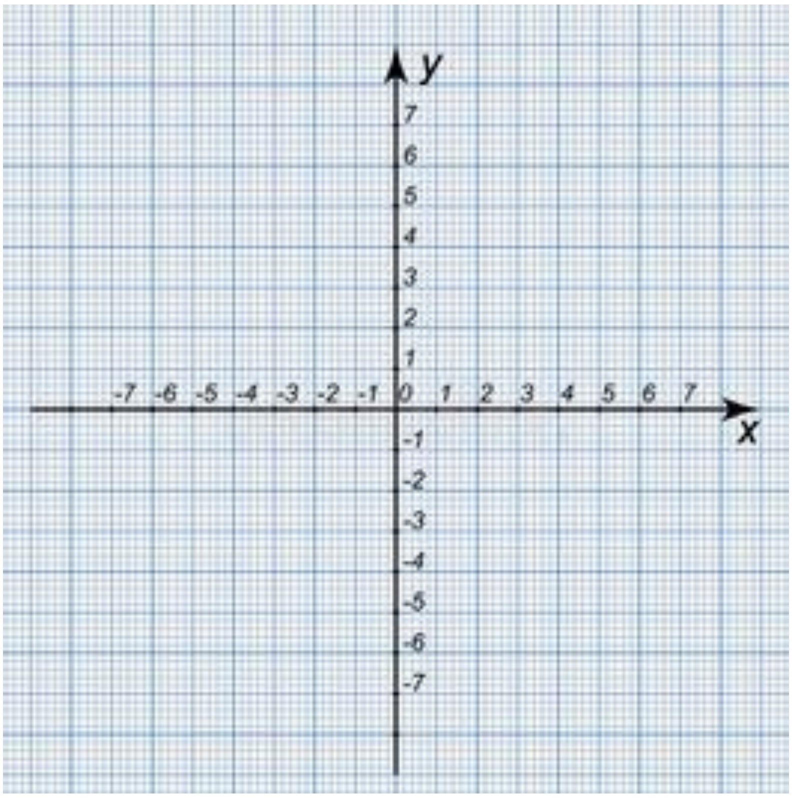
Example9.

Graph the solution set of the following simultaneous inequalities.

$$\begin{cases} x+y-2<0 \\ -x+2y-1>0 \end{cases}$$



<資料來源>

1. **Reflection of points**
   https://www.mathsisfun.com/geometry/reflection.html
   https://byjus.com/maths/reflection/
   https://www.khanacademy.org/math/geometry-home/geometry-coordinate-plane/geometry-reflect-coord-plane/v/reflecting-points-exercise

2. **Half-plane and two-variable inequalities**
   https://saylordotorg.github.io/text_intermediate-algebra/s05-07-solving-inequalities-with-two-.html

3. **Pearson Edexcel AS and A level Mathematics Pure Mathematics Year 1/AS Chapter**

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